

PERIODIC ANALOGS OF WIGNER TRANSFORMS AND WEYL TRANSFORMS

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Abstract. The periodic Wigner transform is introduced. We show that most of the properties of the Euclidean Wigner transform are satisfied in this new setting. Using the periodic Wigner transform, we define the periodic Weyl transform. L^2 -boundedness of periodic Weyl transforms are investigated. We give a necessary and sufficient condition on the symbol to ensure that the corresponding periodic Weyl transform is a Hilbert-Schmidt operator. We show that the product of two periodic Weyl transforms and the adjoint of a periodic Weyl transform are again periodic Weyl transforms. The connection between pseudo-differential operators on $\mathbb{S}^1$ and periodic Weyl transforms is given.

Keywords: Fourier–Wigner transforms, Wigner transforms, Moyal identity, time and frequency marginal conditions, reconstruction formulas, symbols, Weyl transforms, kernels, Hilbert–Schmidt operators, Weyl calculus

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1 INTRODUCTION

Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Euclidean Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ corresponding to the symbol σ is defined by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi$$

for all functions f and g in $L^2(\mathbb{R}^n)$, where $W(f, g)$ is the Euclidean Wigner transform of f and g defined by

$$W(f, g)(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp, \quad x, \xi \in \mathbb{R}^n.$$

Closely related to the Euclidean Wigner transform $W(f, g)$ of f and g in $L^2(\mathbb{R}^n)$ is the Euclidean Fourier–Wigner transform $V(f, g)$ given by

$$V(f, g)(q, p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq \cdot y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy, \quad q, p \in \mathbb{R}^n.$$

The Euclidean Weyl transforms and Euclidean Wigner transforms have been extensively studied in, for instance, [12].

Weyl transforms on different groups such as the upper half plane, the Heisenberg group, the Poincaré unit disk and $\mathbb{Z}$ are investigated in, respectively, [10], [9], [11], [5] and [8]. Closely related to Weyl transforms are pseudo-differential operators and they have been studied in a lot of literatures on different groups. See, for instance, [2, 4, 7, 13].

Let $\mathbb{S}^1$ be the unit circle with center at the origin. Every function φ on $\mathbb{S}^1$ can be identified with a periodic function φ_T of period $T > 0$, where

$$\varphi_T(\theta) = \varphi(e^{2\pi i\theta/T})$$

for all $\theta \in [-T/2, T/2]$. In this paper, periodic functions with periods π , 2π and 4π are the most important for us.

Let $1 \leq p < \infty$. Then the set of all measurable functions f on $[-T/2, T/2]$ for which

$$\|f\|_{L^p[-T/2, T/2]}^p = \int_{-T/2}^{T/2} |f(\theta)|^p d\theta < \infty$$

is denoted by $L^p[-T/2, T/2]$.

We denote by $L^p(\mathbb{Z})$ the set of all measurable functions F on $\mathbb{Z}$ such that

$$\|F\|_{L^p(\mathbb{Z})}^p = \sum_{n \in \mathbb{Z}} |F(n)|^p < \infty.$$

Let $F \in L^1(\mathbb{Z})$. Then we define the Fourier transform ${}^T\mathcal{F}_\mathbb{Z}F$ of $F \in L^1(\mathbb{Z})$ to be the function on $[-T/2, T/2]$ by

$$({}^T\mathcal{F}_\mathbb{Z}F)(\theta) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \theta / T} F(n), \quad \theta \in [-T/2, T/2].$$

We note that defining the Fourier transform on $\mathbb{Z}$ using the above formula leads to a T -periodic function. If f is a function in $L^2[-T/2, T/2]$, then we define the Fourier transform ${}^T\mathcal{F}_{\mathbb{S}^1}f$ of f to be the function on $\mathbb{Z}$ by

$$({}^T\mathcal{F}_{\mathbb{S}^1}f)(n) = \frac{1}{T} \int_{-T/2}^{T/2} e^{-2\pi i n \theta / T} f(\theta) d\theta, \quad n \in \mathbb{Z}.$$

We note that ${}^T\mathcal{F}_\mathbb{Z} : L^2(\mathbb{Z}) \rightarrow L^2[-T/2, T/2]$ is a surjective isometry with

$${}^T\mathcal{F}_\mathbb{Z} = ({}^T\mathcal{F}_{\mathbb{S}^1})^{-1} = ({}^T\mathcal{F}_{\mathbb{S}^1})^*,$$

where

$$({}^T F_{\mathbb{S}^1})^* : L^2[-T/2, T/2] \rightarrow L^2(\mathbb{Z}),$$

and

$$\| {}^T \mathcal{F}_{\mathbb{Z}} F \|_{L^2[-T/2, T/2]} = \| F \|_{L^2(\mathbb{Z})}, \quad F \in L^2(\mathbb{Z}).$$

For $T = 2\pi$, we denote ${}^T \mathcal{F}_{\mathbb{Z}}$ and ${}^T \mathcal{F}_{\mathbb{S}^1}$ by $\mathcal{F}_{\mathbb{Z}}$ and $\mathcal{F}_{\mathbb{S}^1}$, respectively.

Let T and T' be positive numbers. Then for every function H in $L^2([-T/2, T/2] \times \mathbb{Z})$, we define the Fourier transform $({}^{T,T'}) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} H$ of H to be the function on $\mathbb{Z} \times [-T'/2, T'/2]$ by

$$\left(({}^{T,T'}) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} H \right) (m, \theta) = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n \in \mathbb{Z}} e^{-(2\pi i m \varphi / T') + (2\pi i n \theta / T)} H(\varphi, n) d\varphi$$

for all (m, θ) in $\mathbb{Z} \times [-T'/2, T'/2]$. We note that $({}^{T,T'}) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} H$ is T' -periodic with respect to the second variable. Similarly, we can define the Fourier transform $({}^{T',T}) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}}$ of a function in $L^2(\mathbb{Z} \times [-T/2, T/2])$.

Let $1 \leq p < \infty$. Then we define $L^p(\mathbb{Z} \times [-T/2, T/2])$ to be the space of all measurable functions h on $\mathbb{Z} \times [-T/2, T/2]$ such that

$$\| h \|_{L^p(\mathbb{Z} \times [-T/2, T/2])}^p = \sum_{n \in \mathbb{Z}} \int_{-T/2}^{T/2} |h(n, \theta)|^p d\theta < \infty.$$

Similarly, we can define $L^p([-T/2, T/2] \times \mathbb{Z})$.

In Section 2, we define the periodic Fourier–Wigner transform and the periodic Wigner transform. We show that they both satisfy the Moyal identity. We give a formula to reconstruct a function from its periodic Wigner transform up to a constant phase factor. Then we give the time and frequency marginal conditions, and a convolution theorem for the periodic Wigner transform. In Section 3, by means of the periodic Wigner transform, we define the periodic Weyl transform. We give a necessary and sufficient condition on the symbol for its periodic Weyl transform to be first self-adjoint and then Hilbert–Schmidt. We also give a reconstruction formula for retrieving the symbol from its periodic Weyl transform. In Section 4, we prove that the adjoints and the product of periodic Weyl transforms are also periodic Weyl transforms. In Section 5, we give the connection between periodic Weyl transforms and pseudo-differential operators on $\mathbb{S}^1$.

2 PERIODIC FOURIER–WIGNER TRANSFORMS

Let $f \in L^2[-\pi, \pi]$. Then for all $(n, \theta) \in \mathbb{Z} \times [-\pi, \pi]$, we define $\rho(n, \theta)f$ on $[-\pi, \pi]$ by

$$(\rho(n, \theta)f)(\varphi) = e^{in(\varphi + (\theta/2))} f(\theta + \varphi)$$

for all $\varphi \in [-\pi, \pi]$. We note that $\rho(\theta, n) : L^2[-\pi, \pi] \rightarrow L^2[-\pi, \pi]$ is a unitary operator and

$$\rho(n, \theta)^* = \rho(-n, -\theta)$$

for all $(n, \theta) \in \mathbb{Z} \times [-\pi, \pi]$. In fact, for all f and g in $L^2[-\pi, \pi]$,

$$\begin{aligned} (\rho(n, \theta)f, g)_{L^2[-\pi, \pi]} &= \int_{-\pi}^{\pi} (\rho(n, \theta)f)(\varphi) \overline{g(\varphi)} d\varphi \\ &= \int_{-\pi}^{\pi} e^{in(\varphi + (\theta/2))} f(\theta + \varphi) \overline{g(\varphi)} d\varphi \end{aligned}$$

for all $(n, \theta) \in \mathbb{Z} \times [-\pi, \pi]$. Let $\gamma = \theta + \varphi$. Then

$$\begin{aligned} (\rho(n, \theta)f, g)_{L^2[-\pi, \pi]} &= \int_{-\pi+\theta}^{\pi+\theta} e^{in(\gamma - (\theta/2))} f(\gamma) \overline{g(\gamma - \theta)} d\gamma \quad (2.1) \\ &= \int_{-\pi}^{\pi} e^{in(\gamma - (\theta/2))} f(\gamma) \overline{g(\gamma - \theta)} d\gamma \\ &= \int_{-\pi}^{\pi} f(\gamma) \overline{(\rho(-n, -\theta)g)(\gamma)} d\gamma, \end{aligned}$$

where in (2.1) we use the fact that the integrand is 2π -periodic with respect to φ and hence the integrals over any interval of length 2π are equal.

For all f and g in $L^2[-\pi, \pi]$, we define the periodic Fourier–Wigner transform $V(f, g)$ of f and g to be the function on $\mathbb{Z} \times [-2\pi, 2\pi]$ by

$$V(f, g)(n, \theta) = (\rho(n, \theta)f, g)_{L^2[-\pi, \pi]}, \quad (n, \theta) \in \mathbb{Z} \times [-2\pi, 2\pi].$$

Therefore

$$V(f, g)(n, \theta) = \int_{-\pi}^{\pi} e^{in(\varphi + (\theta/2))} f(\theta + \varphi) \overline{g(\varphi)} d\varphi$$

for all $(n, \theta) \in \mathbb{Z} \times [-2\pi, 2\pi]$. It is then clear that $V(f, g)(n, \theta)$ is 4π -periodic with respect to θ . Let $\gamma = \varphi + \frac{\theta}{2}$. Then

$$V(f, g)(n, \theta) = \int_{-\pi+\frac{\theta}{2}}^{\pi+\frac{\theta}{2}} e^{in\gamma} f\left(\gamma + \frac{\theta}{2}\right) \overline{g\left(\gamma - \frac{\theta}{2}\right)} d\gamma.$$

Since the integrand is 2π -periodic with respect to γ , it follows that

$$V(f, g)(n, \theta) = \int_{-\pi}^{\pi} e^{in\gamma} f\left(\gamma + \frac{\theta}{2}\right) \overline{g\left(\gamma - \frac{\theta}{2}\right)} d\gamma, \quad (n, \theta) \in \mathbb{Z} \times [-2\pi, 2\pi].$$

In fact, for all $(n, \theta) \in \mathbb{Z} \times [-2\pi, 2\pi]$,

$$V(f, g)(n, \theta) = 2\pi \mathcal{F}_{\mathbb{S}^1} \left(f\left(\cdot + \frac{\theta}{2}\right) \overline{g\left(\cdot - \frac{\theta}{2}\right)} \right) (-n). \quad (2.2)$$

We have the following Moyal identity for the periodic Fourier–Wigner transform.

Theorem 2.1 *Let f_1, f_2, g_1 and g_2 be functions in $L^2[-\pi, \pi]$. Then*

$$(V(f_1, g_1), V(f_2, g_2))_{L^2(\mathbb{Z} \times [-2\pi, 2\pi])} = (f_1, f_2)_{L^2[-\pi, \pi]} \overline{(g_1, g_2)_{L^2[-\pi, \pi]}}.$$

Proof By (2.2), we have

$$\begin{aligned} & (V(f_1, g_1), V(f_2, g_2))_{L^2(\mathbb{Z} \times [-2\pi, 2\pi])} \\ &= \sum_{n \in \mathbb{Z}} \int_{-2\pi}^{2\pi} V(f_1, g_1)(n, \theta) \overline{V(f_2, g_2)(n, \theta)} d\theta \\ &= 4\pi^2 \sum_{n \in \mathbb{Z}} \int_{-2\pi}^{2\pi} \mathcal{F}_{\mathbb{S}^1} \left(f_1\left(\cdot + \frac{\theta}{2}\right) \overline{g_1\left(\cdot - \frac{\theta}{2}\right)} \right) (n) \\ & \quad \overline{\mathcal{F}_{\mathbb{S}^1} \left(f_2\left(\cdot + \frac{\theta}{2}\right) \overline{g_2\left(\cdot - \frac{\theta}{2}\right)} \right) (n)} d\theta. \end{aligned}$$

By the Parseval identity, we get

$$\begin{aligned} & (V(f_1, g_1), V(f_2, g_2))_{L^2(\mathbb{Z} \times [-2\pi, 2\pi])} \\ &= 2\pi \int_{-2\pi}^{2\pi} \int_{-\pi}^{\pi} f_1\left(\varphi + \frac{\theta}{2}\right) \overline{g_1\left(\varphi - \frac{\theta}{2}\right)} f_2\left(\varphi + \frac{\theta}{2}\right) \overline{g_2\left(\varphi - \frac{\theta}{2}\right)} d\varphi d\theta. \end{aligned}$$

Making a change of variables from (θ, φ) to (x, y) by $x = \varphi + \frac{\theta}{2}$ and $y = \varphi - \frac{\theta}{2}$, we see that the rectangle $\{(\theta, \varphi) : -2\pi \leq \theta \leq 2\pi, -\pi \leq \varphi \leq \pi\}$ in the $\theta\varphi$ -plane becomes the square S with vertices $(-2\pi, 0)$, $(0, -2\pi)$, $(2\pi, 0)$ and $(0, 2\pi)$ in the xy -plane. Let h be the function on the square S defined by

$$h(x, y) = 2\pi f_1(x) \overline{g_1(y)} f_2(x) \overline{g_2(y)}, \quad (x, y) \in S.$$

Then

$$\begin{aligned}
& (V(f_1, g_1), V(f_2, g_2))_{L^2(\mathbb{Z} \times [-2\pi, 2\pi])} \\
&= \int_{-2\pi}^0 \int_{-2\pi-x}^{2\pi+x} h(x, y) dy dx + \int_0^{2\pi} \int_{-2\pi+x}^{2\pi-x} h(x, y) dy dx \\
&= \int_{-2\pi}^0 \left\{ \int_{-2\pi-x}^{2\pi-x} h(x, y) dy - \int_{2\pi+x}^{2\pi-x} h(x, y) dy \right\} dx \\
&+ \int_0^{2\pi} \left\{ \int_{-2\pi+x}^{2\pi+x} h(x, y) dy - \int_{2\pi-x}^{2\pi+x} h(x, y) dy \right\} dx.
\end{aligned}$$

Since h is 4π -periodic with respect to y , we get

$$\begin{aligned}
& (V(f_1, g_1), V(f_2, g_2))_{L^2(\mathbb{Z} \times [-2\pi, 2\pi])} \\
&= \int_{-2\pi}^0 \int_{-2\pi}^{2\pi} h(x, y) dy dx - \int_{-2\pi}^0 \int_{2\pi+x}^{2\pi-x} h(x, y) dy dx \\
&+ \int_0^{2\pi} \int_{-2\pi}^{2\pi} h(x, y) dy dx - \int_0^{2\pi} \int_{2\pi-x}^{2\pi+x} h(x, y) dy dx \\
&= \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} h(x, y) dx dy \\
&- \int_{-2\pi}^0 \int_{2\pi+x}^{2\pi-x} h(x, y) dy dx - \int_0^{2\pi} \int_{2\pi-x}^{2\pi+x} h(x, y) dy dx \\
&= \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} h(x, y) dx dy \\
&- \int_{-2\pi}^0 \int_{2\pi+x}^{2\pi-x} h(x, y) dy dx - \int_0^{2\pi} \int_{2\pi-x}^{2\pi+x} h(x, y) dy dx. \quad (2.3)
\end{aligned}$$

Using the change of variables $\tilde{x} = x - 2\pi$ and $\tilde{y} = y - 2\pi$ in, respectively,

the second integral and the first integral, we get

$$\begin{aligned}
& - \int_{-2\pi}^0 \int_{2\pi+x}^{2\pi-x} h(x, y) dy dx - \int_0^{2\pi} \int_{2\pi-x}^{2\pi+x} h(x, y) dy dx \\
= & - \int_{-2\pi}^0 \int_x^{-x} h(x, \tilde{y} + 2\pi) d\tilde{y} dx - \int_{-2\pi}^0 \int_{-x}^{4\pi+x} h(x, y) dy dx \\
= & - \int_{-2\pi}^0 \left\{ \int_x^{-x} h(x, y) dy + \int_{-x}^{4\pi+x} h(x, y) dy \right\} dx \\
= & - \int_{-2\pi}^0 \left\{ \int_x^{4\pi+x} h(x, y) dy \right\} dx \\
= & - \int_{-2\pi}^0 \int_0^{4\pi} h(x, y) dy dx. \tag{2.4}
\end{aligned}$$

Therefore by (2.3) and (2.4), we get

$$(V(f_1, g_1), V(f_2, g_2))_{L^2(\mathbb{Z} \times [-2\pi, 2\pi])} = (f_1, f_2)_{L^2[-\pi, \pi]} \overline{(g_1, g_2)_{L^2[-\pi, \pi]}}.$$

□

3 PERIODIC WIGNER TRANSFORMS

Let f and g be functions in $L^2[-\pi, \pi]$. Then we define the periodic Wigner transform $W(f, g)$ of f and g by

$$W(f, g) = \left({}^{(2\pi, 4\pi)}\mathcal{F}_{\mathbb{Z} \times \mathbb{S}^1} \right) V(f, g). \tag{3.1}$$

By (2.2), for all $(\varphi, m) \in [-\pi, \pi] \times \mathbb{Z}$, we get

$$\begin{aligned}
W(f, g)(\varphi, m) &= {}^{4\pi}\mathcal{F}_{\mathbb{S}^1} \left(f \left(\varphi + \frac{\cdot}{2} \right) \overline{g \left(\varphi - \frac{\cdot}{2} \right)} \right) (m) \\
&= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{-im\theta/2} f \left(\varphi + \frac{\theta}{2} \right) \overline{g \left(\varphi - \frac{\theta}{2} \right)} d\theta.
\end{aligned} \tag{3.2}$$

Note that $W(f, g)$ is 2π -periodic with respect to φ . As in the case of the periodic Fourier–Wigner transform, we have the following Moyal identity for the periodic Wigner transform.

Theorem 3.1 *Let f_1, f_2, g_1 and g_2 be functions in $L^2[-\pi, \pi]$. Then*

$$(W(f_1, g_1), W(f_2, g_2))_{L^2([-\pi, \pi] \times \mathbb{Z})} = (f_1, f_2)_{L^2[-\pi, \pi]} \overline{(g_1, g_2)_{L^2[-\pi, \pi]}}.$$

Proof By (3.1), the Parseval identity and the Moyal identity for the periodic Fourier–Wigner transform, we get

$$\begin{aligned} & (W(f_1, g_1), W(f_2, g_2))_{L^2([-\pi, \pi] \times \mathbb{Z})} \\ &= \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} W(f_1, g_1)(\varphi, m) \overline{W(f_2, g_2)(\varphi, m)} d\varphi \\ &= \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} \left((2\pi, 4\pi) \mathcal{F}_{\mathbb{Z} \times \mathbb{S}^1} V \right) (f_1, g_1)(\varphi, m) \\ & \quad \overline{\left((2\pi, 4\pi) \mathcal{F}_{\mathbb{Z} \times \mathbb{S}^1} V \right) (f_2, g_2)(\varphi, m)} d\varphi \\ &= \sum_{n \in \mathbb{Z}} \int_{-2\pi}^{2\pi} V(f_1, g_1)(n, \theta) \overline{V(f_2, g_2)(n, \theta)} d\theta \\ &= (f_1, f_2)_{L^2[-\pi, \pi]} \overline{(g_1, g_2)_{L^2[-\pi, \pi]}}. \end{aligned}$$

□

As in the case of the Euclidean Wigner transform, the following proposition guarantees that for all $f \in L^2[-\pi, \pi]$, $W(f, f)$ is real.

Proposition 3.2 *Let f and g be in $L^2[-\pi, \pi]$. Then*

$$W(f, g) = \overline{W(g, f)}.$$

In particular, $W(f, f)$ is a real-valued function on $[-\pi, \pi] \times \mathbb{Z}$.

For simplicity, we denote $W(f, f)$ by $W(f)$. The following theorem states that we can reconstruct the original signal f from its periodic Wigner transform up to a constant phase factor.

Theorem 3.3 *Let $f \in L^2[-\pi, \pi]$. Then for all $\theta \in [-\pi, \pi]$,*

$$f(\theta) \overline{f(0)} = \sum_{m \in \mathbb{Z}} e^{im\theta/2} W(f) \left(\frac{\theta}{2}, m \right).$$

Proof By (3.2), for all θ and φ in $[-\pi, \pi]$,

$$f\left(\varphi + \frac{\theta}{2}\right) \overline{f\left(\varphi - \frac{\theta}{2}\right)} = \left({}^4\mathcal{F}_{\mathbb{Z}}(W(f)(\varphi, \cdot))\right)(\theta) = \sum_{m \in \mathbb{Z}} e^{im\theta/2} W(f)(\varphi, m). \quad (3.3)$$

By letting $\varphi = \frac{\theta}{2}$, we get

$$f(\theta) \overline{f(0)} = \sum_{m \in \mathbb{Z}} e^{im\theta/2} W(f)\left(\frac{\theta}{2}, m\right).$$

□

We have the time and frequency marginal conditions for the periodic Wigner transform.

Proposition 3.4 *Let $f \in L^2[-\pi, \pi]$.*

(i) *For all $\varphi \in [-\pi, \pi]$,*

$$|f(\varphi)|^2 = \sum_{m \in \mathbb{Z}} W(f)(\varphi, m).$$

(ii) *For all $m \in \mathbb{Z}$,*

$$|(\mathcal{F}_{\mathbb{S}^1} f)(m)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(f)(\varphi, 2m) d\varphi.$$

Proof By letting $\theta = 0$ in (3.3), we get Part (i). To prove Part (ii), let $m \in \mathbb{Z}$. Then

$$\int_{-\pi}^{\pi} W(f)(\varphi, 2m) d\varphi = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-2\pi}^{2\pi} e^{-im\theta} f\left(\varphi + \frac{\theta}{2}\right) \overline{f\left(\varphi - \frac{\theta}{2}\right)} d\theta d\varphi.$$

Changing variables from (θ, φ) to (x, y) by letting $x = \varphi + \frac{\theta}{2}$ and $y = \varphi - \frac{\theta}{2}$, the rectangle $\{(\varphi, \theta) : -\pi \leq \varphi \leq \pi, -2\pi \leq \theta \leq 2\pi\}$ in the $\varphi\theta$ -plane becomes the square S with vertices $(0, -2\pi)$, $(2\pi, 0)$, $(0, 2\pi)$, $(-2\pi, 0)$ in the xy -plane. Let h be the function on S defined by

$$h(x, y) = e^{-im(x-y)} f(x) \overline{f(y)}, \quad (x, y) \in S.$$

Then

$$\begin{aligned}
& \int_{-\pi}^{\pi} W(f)(\varphi, 2m) d\varphi \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} h(x, y) dx dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-im(x-y)} f(x) \overline{f(y)} dx dy \\
&= 2\pi |(\mathcal{F}_{\mathbb{S}^1} f)(m)|^2.
\end{aligned}$$

□

The following proposition gives the shift invariance of the periodic Wigner transform and the proof is straightforward.

Proposition 3.5 *Let*

$$f \in L^2[-\pi, \pi], \quad \eta \in [-\pi, \pi].$$

For $k \in \mathbb{Z}$, *let*

$$g(\gamma) = e^{ik\gamma/2} f(\gamma - \eta), \quad \gamma \in [-\pi, \pi].$$

Then

$$W(g)(\varphi, m) = W(f)(\varphi - \eta, m - k)$$

for all $(\varphi, m) \in [-\pi, \pi] \times \mathbb{Z}$.

Like the Euclidean Fourier transform, we have the convolution theorem for the periodic Wigner transform.

Proposition 3.6 *Let* f *and* g *be functions in* $L^2[-\pi, \pi]$. *Then*

$$W(fg)(\varphi, m) = \left(W(f)(\varphi, \cdot) * W(g)(\varphi, \cdot) \right)(m), \quad (\varphi, m) \in [-\pi, \pi] \times \mathbb{Z},$$

where $*$ *is the convolution on* $\mathbb{Z}$ *defined by*

$$(F * G)(m) = \sum_{n \in \mathbb{Z}} F(m - n) G(n), \quad m \in \mathbb{Z},$$

for all F *and* G *in* $L^2(\mathbb{Z})$.

Proof It is enough to show that

$${}^{4\pi}\mathcal{F}_{\mathbb{Z}}(W(fg)(\varphi, \cdot))(\theta) = {}^{4\pi}\mathcal{F}_{\mathbb{Z}}(W(f)(\varphi, \cdot))(\theta) {}^{4\pi}\mathcal{F}_{\mathbb{Z}}(W(g)(\varphi, \cdot))(\theta)$$

for all $\varphi \in [-\pi, \pi]$ and $\theta \in [-2\pi, 2\pi]$. Using (3.2), we get the result.

□

4 PERIODIC WEYL TRANSFORMS

Let σ be a function in $L^2([-\pi, \pi] \times \mathbb{Z})$. Let f and g be functions in $L^2[-\pi, \pi]$. Then we define the periodic Weyl transform corresponding to the symbol σ by

$$(W_\sigma f, g)_{L^2[-\pi, \pi]} = \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} \sigma(\varphi, m) W(f, g)(\varphi, m) d\varphi.$$

By the Parseval identity and (3.1), we have

$$\begin{aligned} & (W_\sigma f, g)_{L^2[-\pi, \pi]} \\ &= \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} \sigma(\varphi, m) \left((2\pi, 4\pi) \mathcal{F}_{\mathbb{Z} \times \mathbb{S}^1} V(f, g) \right) (\varphi, m) d\varphi \\ &= \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \int_{-2\pi}^{2\pi} \left((4\pi, 2\pi) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} \sigma \right) (n, \theta) V(f, g)(n, \theta) d\theta \\ &= \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \int_{-2\pi}^{2\pi} \left((4\pi, 2\pi) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} \sigma \right) (n, \theta) (\rho(n, \theta) f, g)_{L^2[-\pi, \pi]} d\theta. \end{aligned}$$

Hence

$$(W_\sigma f)(\gamma) = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \int_{-2\pi}^{2\pi} \left((4\pi, 2\pi) \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} \sigma \right) (n, \theta) (\rho(n, \theta) f)(\gamma) d\theta. \quad (4.1)$$

By the definition of the periodic Wigner transform,

$$\begin{aligned} & (W_\sigma f, g)_{L^2[-\pi, \pi]} \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} \sigma(\varphi, m) \int_{-2\pi}^{2\pi} e^{-im\theta} f\left(\varphi + \frac{\theta}{2}\right) \overline{g\left(\varphi - \frac{\theta}{2}\right)} d\theta d\varphi. \end{aligned}$$

By the change of variables from φ to $\gamma = \varphi - \frac{\theta}{2}$, we get

$$\begin{aligned} & (W_\sigma f, g)_{L^2[-\pi, \pi]} \\ &= \frac{1}{4\pi} \sum_{m \in \mathbb{Z}} \int_{-2\pi}^{2\pi} \int_{-\pi - \frac{\theta}{2}}^{\pi - \frac{\theta}{2}} \sigma\left(\gamma + \frac{\theta}{2}, m\right) e^{-im\theta} f(\gamma + \theta) \overline{g(\gamma)} d\gamma d\theta \\ &= \frac{1}{4\pi} \sum_{m \in \mathbb{Z}} \int_{-2\pi}^{2\pi} \int_{-\pi}^{\pi} \sigma\left(\gamma + \frac{\theta}{2}, m\right) e^{-im\theta} f(\gamma + \theta) \overline{g(\gamma)} d\gamma d\theta \\ &= \int_{-\pi}^{\pi} \left\{ \frac{1}{4\pi} \sum_{m \in \mathbb{Z}} \int_{-2\pi}^{2\pi} e^{-im\theta} \sigma\left(\gamma + \frac{\theta}{2}, m\right) f(\gamma + \theta) d\theta \right\} \overline{g(\gamma)} d\gamma. \end{aligned}$$

Therefore

$$\begin{aligned}
(W_\sigma f)(\gamma) &= \frac{1}{4\pi} \sum_{m \in \mathbb{Z}} \int_{-2\pi}^{2\pi} e^{-im\theta} \sigma\left(\gamma + \frac{\theta}{2}, m\right) f(\gamma + \theta) d\theta \\
&= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} (\mathcal{F}_{2,\mathbb{Z}}\sigma)\left(\gamma + \frac{\theta}{2}, -\theta\right) f(\gamma + \theta) d\theta \quad (4.2)
\end{aligned}$$

for all $\gamma \in [-\pi, \pi]$, where $\mathcal{F}_{2,\mathbb{Z}}\sigma$ is the Fourier transform of σ with respect to the second variable. By the change of variables from θ to $\eta = \gamma + \theta$, we get

$$(W_\sigma f)(\gamma) = \frac{1}{4\pi} \int_{\gamma-2\pi}^{\gamma+2\pi} (\mathcal{F}_{2,\mathbb{Z}}\sigma)\left(\frac{\eta+\gamma}{2}, \gamma-\eta\right) f(\eta) d\eta, \quad \gamma \in [-\pi, \pi].$$

But $\mathcal{F}_{2,\mathbb{Z}}\sigma$ is 2π -periodic with respect to the first variable and the second variable and hence 4π -periodic with respect to both. Therefore

$$(W_\sigma f)(\gamma) = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} (\mathcal{F}_{2,\mathbb{Z}}\sigma)\left(\frac{\eta+\gamma}{2}, \gamma-\eta\right) f(\eta) d\eta, \quad \gamma \in [-\pi, \pi].$$

Let

$$k_\sigma(\gamma, \eta) = \frac{1}{4\pi} (\mathcal{F}_{2,\mathbb{Z}}\sigma)\left(\frac{\eta+\gamma}{2}, \gamma-\eta\right), \quad \gamma \in [-\pi, \pi], \eta \in [-2\pi, 2\pi]. \quad (4.3)$$

Then W_σ is an integral operator with kernel k_σ , i.e.,

$$(W_\sigma f)(\gamma) = \int_{-2\pi}^{2\pi} k_\sigma(\gamma, \eta) f(\eta) d\eta, \quad \gamma \in [-\pi, \pi]. \quad (4.4)$$

We can easily check that k_σ is 4π -periodic with respect to both variables. It is also important to mention that $W_\sigma f$ is periodic with period 2π by (2.1).

Proposition 4.1 *Let $\sigma \in L^2([-\pi, \pi] \times \mathbb{Z})$. Then $W_\sigma : L^2[-\pi, \pi] \rightarrow L^2[-\pi, \pi]$ is a bounded linear operator and*

$$\|W_\sigma\|_* \leq \frac{1}{4\pi} \|\sigma\|_{L^2([-\pi, \pi] \times \mathbb{Z})}$$

where $\|\cdot\|_*$ is the norm in the C^* -algebra of all bounded linear operators on $L^2[-\pi, \pi]$.

Proof Let f be in $L^2[-\pi, \pi]$. Let $\theta = \gamma - \eta$ and $\varphi = \frac{\eta + \gamma}{2}$. Then using the same computations as in Theorem 2.1 and the Parseval identity, we get

$$\begin{aligned}
& \int_{-2\pi}^{2\pi} \int_{-\pi}^{\pi} |k_{\sigma}(\gamma, \eta)|^2 d\gamma d\eta \\
&= (4\pi)^{-2} \int_{-2\pi}^{2\pi} \int_{-\pi}^{\pi} \left| (\mathcal{F}_{2, \mathbb{Z}} \sigma) \left(\frac{\eta + \gamma}{2}, \gamma - \eta \right) \right|^2 d\gamma d\eta \\
&= 2(4\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |(\mathcal{F}_{2, \mathbb{Z}} \sigma)(\varphi, \theta)|^2 d\theta d\varphi \\
&= 2(4\pi)^{-2} \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} |\sigma(\varphi, n)|^2 d\varphi \\
&= 2(4\pi)^{-2} \|\sigma\|_{L^2([-\pi, \pi] \times \mathbb{Z})}^2.
\end{aligned} \tag{4.5}$$

So,

$$\begin{aligned}
& \|W_{\sigma} f\|_{L^2[-\pi, \pi]} \\
&= \left\{ \int_{-\pi}^{\pi} \left| \int_{-2\pi}^{2\pi} k_{\sigma}(\gamma, \eta) f(\eta) d\eta \right|^2 d\gamma \right\}^{1/2} \\
&\leq \int_{-2\pi}^{2\pi} \left\{ \int_{-\pi}^{\pi} |k_{\sigma}(\gamma, \eta)|^2 |f(\eta)|^2 d\gamma \right\}^{1/2} d\eta \\
&\leq \left\{ \int_{-2\pi}^{2\pi} \int_{-\pi}^{\pi} |k_{\sigma}(\gamma, \eta)|^2 d\gamma d\eta \right\}^{1/2} \|f\|_{L^2[-\pi, \pi]} \\
&\leq \frac{\sqrt{2}}{4\pi} \|\sigma\|_{L^2([-\pi, \pi] \times \mathbb{Z})} \|f\|_{L^2[-2\pi, 2\pi]}.
\end{aligned}$$

□

Theorem 4.2 Let $A : L^2[-\pi, \pi] \rightarrow L^2[-\pi, \pi]$ be a Hilbert-Schmidt operator on $L^2[-\pi, \pi]$ with kernel in $L^2([-\pi/2, \pi/2] \times [-\pi/2, \pi/2])$. Then there exists a unique $\sigma \in L^2([-\pi, \pi] \times \mathbb{Z})$ such that $A = W_{\sigma}$.

Proof Let $k \in L^2([-2\pi, 2\pi] \times [-2\pi, 2\pi])$ be such that

$$(Af)(\gamma) = \frac{\sqrt{2}}{4\pi} \int_{-2\pi}^{2\pi} k(\gamma, \eta) f(\eta) d\eta, \quad \gamma \in [-2\pi, 2\pi],$$

for all $f \in L^2[-2\pi, 2\pi]$. Let $\sigma \in L^2([-\pi, \pi] \times \mathbb{Z})$ be such that

$$k(\gamma, \eta) = (\mathcal{F}_{2, \mathbb{Z}} \sigma) \left(\frac{\eta + \gamma}{2}, \gamma - \eta \right), \quad \gamma, \eta \in [-2\pi, 2\pi].$$

Then the proof is complete if we let

$$\sigma(\varphi, m) = \left((\mathcal{F}_{\mathbb{S}^1} k) \left(\varphi + \frac{\cdot}{2}, \varphi - \frac{\cdot}{2} \right) \right) (m)$$

for all $(\varphi, m) \in [-\pi, \pi] \times \mathbb{Z}$. $\square$

Remark 4.3 Note that in Theorem 4.2, in order to reconstruct σ from the kernel k using $\mathcal{F}_{\mathbb{S}^1}$, we assume that k is π -periodic with respect to both variables.

The following theorem is an inversion formula for periodic Weyl transforms. It shows how we can reconstruct the symbol from the corresponding periodic Weyl transform. For Weyl transforms in other settings, the corresponding formulas and other related formulas can be found in [1, 3, 8].

Theorem 4.4 *Let $\sigma \in L^2([-\pi, \pi] \times \mathbb{Z})$ be such that $\rho(n, \theta)W_\sigma$ is a trace class operator on $L^2[-2\pi, 2\pi]$ for all $(\theta, n) \in [-\pi, \pi] \times \mathbb{Z}$. Then for all $(n, \theta) \in \mathbb{Z} \times [-\pi, \pi]$, we have*

$$(\mathcal{F}_{[-\pi, \pi] \times \mathbb{Z}} \sigma)(n, \theta) = \text{tr}(\rho(-n, \theta)W_\sigma).$$

Proof Let $f \in L^2[-2\pi, 2\pi]$. Then for all $\varphi \in [-2\pi, 2\pi]$,

$$\begin{aligned} & (\rho(n, \theta)W_\sigma f)(\varphi) \\ &= e^{in(\varphi + (\theta/2))} (W_\sigma f)(\theta + \varphi) \\ &= \frac{1}{4\pi} e^{in(\varphi + (\theta/2))} \int_{-2\pi}^{2\pi} k_\sigma(\theta + \varphi, \eta) f(\eta) d\eta. \end{aligned}$$

For all φ and η in $[-2\pi, 2\pi]$, let $k^\theta(\varphi, \eta) = e^{in(\varphi + (\theta/2))} k_\sigma(\theta + \varphi, \eta)$. Then

$$\begin{aligned} & \text{tr}(\rho(n, \theta)W_\sigma) \\ &= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} k^\theta(\varphi, \varphi) d\varphi \\ &= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{in(\varphi + (\theta/2))} k_\sigma(\theta + \varphi, \varphi) d\varphi \\ &= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} e^{in(\varphi + (\theta/2))} (\mathcal{F}_{2, \mathbb{Z}} \sigma) \left(\varphi + \frac{\theta}{2}, \theta \right) d\varphi \end{aligned}$$

for all $(n, \theta) \in \mathbb{Z} \times [-\pi, \pi]$. If we change the variable φ to $\tilde{\varphi}$ by $\tilde{\varphi} = \varphi + \frac{\theta}{2}$ and use the 2π -periodicity of $\mathcal{F}_{2, \mathbb{Z}}\sigma$, we get for all (n, θ) in $\mathbb{Z} \times [-\pi, \pi]$,

$$\begin{aligned} & \text{tr}(\rho(n, \theta)W_\sigma) \\ &= \frac{1}{4\pi} \int_{-2\pi+\theta/2}^{2\pi+\theta/2} e^{in\tilde{\varphi}} (\mathcal{F}_{2, \mathbb{Z}}\sigma)(\tilde{\varphi}, \theta) d\tilde{\varphi} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\varphi} (\mathcal{F}_{2, \mathbb{Z}}\sigma)(\varphi, \theta) d\varphi \\ &= (\mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}}\sigma)(-n, \theta). \end{aligned}$$

□

5 The WEYL CALCULUS

Theorem 5.1 *Let $\sigma \in L^2([-\pi, \pi] \times \mathbb{Z})$. Then*

$$W_\sigma^* = W_{\bar{\sigma}},$$

where W_σ^* is the adjoint of $W_\sigma : L^2[-2\pi, 2\pi] \rightarrow L^2[-2\pi, 2\pi]$. In particular, W_σ is self-adjoint if and only if σ is real-valued.

Proof Let k_σ and k_σ^* be the kernels of W_σ and W_σ^* , respectively. Then it is easy to see that for all $(\gamma, \eta) \in [-2\pi, 2\pi] \times [-2\pi, 2\pi]$,

$$k_\sigma^*(\eta, \gamma) = \overline{k_\sigma(\gamma, \eta)}.$$

Using (4.3), we get

$$k_\sigma^*(\eta, \gamma) = \overline{(\mathcal{F}_{2, \mathbb{Z}}\sigma)\left(\frac{\eta + \gamma}{2}, \gamma - \eta\right)} = (\mathcal{F}_{2, \mathbb{Z}}\bar{\sigma})\left(\frac{\eta + \gamma}{2}, \gamma - \eta\right)$$

for all η and γ in $[-2\pi, 2\pi]$. Using (4.3) again, we see that $k_\sigma^* = k_{\bar{\sigma}}$, where $k_{\bar{\sigma}}$ is the kernel of the Weyl transform corresponding to the symbol $\bar{\sigma}$. □

The following theorem guarantees that the product of two periodic Weyl transforms is still a periodic Weyl transform.

Theorem 5.2 *Let σ and τ be in $L^2([-\pi, \pi] \times \mathbb{Z})$. Then*

$$W_\sigma W_\tau = W_\omega,$$

where ω is defined by

$$\begin{aligned} (\mathcal{F}_{2,\mathbb{Z}}\omega)(\varphi, \theta) &= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} (\mathcal{F}_{2,\mathbb{Z}}\sigma) \left(\frac{\varphi + \eta}{2} + \frac{\theta}{4}, \varphi - \eta + \frac{\theta}{2} \right) \times \\ &\quad (\mathcal{F}_{2,\mathbb{Z}}\tau) \left(\frac{\varphi + \eta}{2} - \frac{\theta}{4}, -\varphi + \eta + \frac{\theta}{2} \right) d\eta. \end{aligned}$$

Proof By (4.3) and Proposition 4.1, W_σ and W_τ are Hilbert-Schmidt operators with kernel k_σ and k_τ respectively. So, their composition is also a Hilbert-Schmidt operator with kernel given by

$$k_\omega(\gamma, \lambda) = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} k_\sigma(\gamma, \eta) k_\tau(\eta, \lambda) d\eta.$$

On the other hand, by Theorem 4.2, $W_\sigma W_\tau$ is a periodic Weyl transform. Hence by (4.3) and change of variables from (λ, γ) to (φ, θ) via $\theta = \gamma - \lambda$ and $\varphi = \frac{\lambda + \gamma}{2}$, we get the result. $\square$

6 THE CONNECTION WITH PSEUDO-DIFFERENTIAL OPERATORS

Let $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Then we define the pseudo-differential operator corresponding to the symbol σ on $L^2(\mathbb{S}^1)$ by

$$(T_\sigma f)(\varphi) = \sum_{m \in \mathbb{Z}} e^{im\theta} \sigma(\varphi, m) (\mathcal{F}_{\mathbb{S}^1} f)(m), \quad f \in L^2(\mathbb{S}^1).$$

Pseudo-differential operators on $\mathbb{S}^1$ have been extensively studied. See, for instance, [6]. The following theorem gives the connection between periodic Weyl transforms and pseudo-differential operators on $\mathbb{S}^1$.

Theorem 6.1 *Let $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Then $T_\sigma = W_\tau$, where $\tau \in L^2(\mathbb{S}^1 \times \mathbb{Z})$ and is given by*

$$\left({}^{(4\pi, 2\pi)} \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} \tau \right) (n, -\theta) = e^{in\theta/2} (\mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} \sigma) (n, \theta), \quad (n, \theta) \in \mathbb{Z} \times \mathbb{S}^1.$$

Proof Let $f \in L^2(\mathbb{S}^1)$. Then

$$\begin{aligned}
(T_\sigma f)(\varphi) &= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{im(\varphi-\eta)} \sigma(\varphi, m) f(\eta) d\eta \\
&= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{-im\theta} \sigma(\varphi, m) f(\varphi + \theta) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathcal{F}_{2, \mathbb{Z}} \sigma)(\varphi, -\theta) f(\varphi + \theta) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} e^{in\varphi} (\mathcal{F}_{1, \mathbb{S}^1} \mathcal{F}_{2, \mathbb{Z}} \sigma)(n, -\theta) f(\varphi + \theta) d\theta \\
&= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} \sum_{n \in \mathbb{Z}} e^{in\varphi} (\mathcal{F}_{1, \mathbb{S}^1} \mathcal{F}_{2, \mathbb{Z}} \sigma)(n, -\theta) f(\varphi + \theta) d\theta \\
&= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\theta/2} (\mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} \sigma)(n, -\theta) (\rho(n, \theta) f)(\varphi) d\theta \\
&= \frac{1}{4\pi} \int_{-2\pi}^{2\pi} \sum_{n \in \mathbb{Z}} \left({}^{(4\pi, 2\pi)} \mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z} \tau} \right) (n, \theta) (\rho(n, \theta) f)(\varphi) d\theta
\end{aligned}$$

for all $\varphi \in [-\pi, \pi]$. So, by (4.1)

$$T_\sigma = W_\tau.$$

□

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