

LOCALIZATION OPERATORS ON THE UNIT DISK

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ABSTRACT. We show that on the unit disk, the localization operators associated to admissible wavelets and separable symbols are paracommutators. We have obtained conditions on the symbol of the localization operator such that it can be expressed in terms of a paraproduct and also under certain other condition on the symbol it is a Fourier multiplier.

KEYWORDS: *Unit disk, square-integrable representations, localization operators, paracommutators, paraproducts, Fourier multipliers*

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1. INTRODUCTION

Localization operators arise in quantum mechanics, signal analysis and partial differential equations. The rationale for the terminology lies in the observation that these operators are used as a mathematical tool to localize a signal on the time frequency plane. The origin of the localization operators can be traced back to the study of a class of bounded linear operators introduced by Daubechies in [3, 4] in signal analysis. The Daubechies operator $D_{F,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ associated with the function $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi \in L^2(\mathbb{R}^{2n})$ with $\|\varphi\|_2 = 1$ is same as the localization operator, $L_{F,\varphi}$, associated to the admissible wavelet φ for the Schrödinger representation of the Weyl-Heisenberg group on $L^2(\mathbb{R}^n)$. Motivated by these operators, localization operators are defined on locally compact groups. Many studies of localization operators have been done extensively on $\mathbb{R}^n$, locally compact groups, affine groups and polar affine groups in [1, 7, 8, 10, 12, 14, 19], to name a few.

For the affine group U , it has been shown in Wong [20] that for any admissible wavelet φ , if the symbol F is of the form

$$F(b, a) = \beta(b)\alpha(a), \quad (b, a) \in U,$$

where α and β are suitable functions on $(0, \infty)$ and $(-\infty, \infty)$, respectively, then the localization operator $L_{F,\varphi}$, is in fact a paracommutator studied by Janson, Peetre, Wong and others in [11, 15, 16, 17, 18, 20]. In [20], Wong has also proved that if F is a suitable function of b only, then $L_{F,\varphi}$ is given in terms of a paraproduct in the sense of Coifman and Meyer [2], and if it is a function of a only, then $L_{F,\varphi}$ is a Fourier multiplier. The affine group has been singled out as an important group on which localization operators are built because the affine group has important connections with complex analysis and hyperbolic geometry. See [18, 21] for the basics of the affine group.

The aim of this paper is to obtain analogs of the results in [20] for the unit disk. These are given in Section 2. The irreducible and unitary representations of the unit disk on the Hardy spaces $H_{\pm}^2(\mathbb{R})$ are also analyzed in Section 2. Admissible wavelets and square integrability for the representations are investigated in Section 3. Localization operators are introduced in Section 4 and it is shown in Section 5 that localization operators are given by paracommutators, and by paraproductions in Section 6, and by Fourier multipliers in Section 7.

2. THE UNIT DISK

Let U be the upper half plane given by

$$U = \{(b, a) : b \in \mathbb{R}, a > 0\}.$$

Then U becomes a group with the binary operation $\cdot_U$ defined by

$$(b, a) \cdot_U (c, d) = (b + ac, ad)$$

for all (b, a) and (c, d) in U . With respect to the multiplication $\cdot_U$, we can verify that U is a non-abelian group. $(\frac{-b}{a}, \frac{1}{a})$ is the inverse element of (b, a) and $(0, 1)$ is the identity element in U . Moreover, U is a non-unimodular Lie group with the left Haar measure $d\mu$ and the right Haar measure dv given by

$$d\mu(b, a) = \frac{db da}{a^2}$$

and

$$dv(b, a) = \frac{db da}{a}.$$

We denote the open unit disk with center at the origin in $\mathbb{R}^2$ by $\mathbb{D}$. The Riemannian metric $(\cdot, \cdot)_z$ on $\mathbb{D}$ is defined by

$$(u, v)_z = \frac{(u, v)}{(1 - |z|^2)^2}, \quad z = (x, y) \in \mathbb{D},$$

for all u and v in $\mathbb{R}^2$, where (u, v) is the Euclidean inner product on $\mathbb{R}^2$. This Riemannian metric on $\mathbb{D}$ gives

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}, \quad z = (x, y) \in \mathbb{D}.$$

The Cayley transform, $c : \mathbb{D} \rightarrow U$ is defined by

$$c(x, y) = \left(\frac{2x}{x^2 + (1 - y)^2}, \frac{1 - (x^2 + y^2)}{x^2 + (1 - y)^2} \right), \quad z = x + iy \in \mathbb{D}.$$

Equipping U with the Riemannian metric

$$\frac{db^2 + da^2}{4a^2}, \quad (b, a) \in U,$$

the Cayley transform c is an isometry of $\mathbb{D}$ onto U . The inverse $c^{-1} : U \rightarrow \mathbb{D}$ is given by

$$c^{-1}(b, a) = \left(\frac{2b}{b^2 + (1 + a)^2}, \frac{a^2 + b^2 - 1}{b^2 + (1 + a)^2} \right), \quad (b, a) \in U.$$

Moreover, it is well known that c is a biholomorphism. so, the group law on $\mathbb{D}$ is defined by

$$z \cdot_{\mathbb{D}} z' = c^{-1} (c(z) \cdot_U c(z'))$$

for all z and z' in $\mathbb{D}$. The identity element in $\mathbb{D}$ is the origin. The left Haar measure and the right Haar measure on the disk are given by

$$d\mu_L(x, y) = \frac{dx dy}{(1 - x^2 - y^2)^2}, \quad (x, y) \in \mathbb{D},$$

and

$$d\mu_R = \frac{dx dy}{(1 - x^2 - y^2)(x^2 + (1 - y)^2)}, \quad (x, y) \in \mathbb{D},$$

respectively. We note that

$$d\mu_L((x, y)^{-1}) = \Delta(x, y)^{-1} d\mu_L(x, y), \quad (x, y) \in \mathbb{D},$$

and the following theorem from [13].

THEOREM 2.1. *Let $f \in L^2(U, d\mu)$. Then*

$$\frac{1}{4} \int_U f(b, a) \frac{db da}{a^2} = \int_{\mathbb{D}} (f \circ c)(p, q) \frac{dp dq}{(1 - p^2 - q^2)^2}.$$

The modular function Δ on $\mathbb{D}$ is given by

$$\Delta(x, y) = \frac{x^2 + (1 - y)^2}{1 - x^2 - y^2}, \quad (x, y) \in \mathbb{D}.$$

A simple calculation gives

$$\Delta(x, y) = \frac{1}{\text{Im}(c(x, y))}, \quad (x, y) \in \mathbb{D}.$$

Then $\Delta : \mathbb{D} \rightarrow (0, \infty)$ is a group homomorphism and its kernel $\text{Ker}(\Delta)$ is given by

$$\text{Ker}(\Delta) = \left\{ c^{-1}(b, 1) : b \in \mathbb{R} \right\} = \left\{ \left(\frac{2b}{b^2 + 4}, \frac{b^2}{b^2 + 4} \right) : b \in \mathbb{R} \right\}.$$

Let $H_+^2(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined by

$$H_+^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq [0, \infty)\},$$

where $\text{supp}(\hat{f})$ is the set of all $\xi \in \mathbb{R}$ for which there is no neighbourhood of ξ on which $\hat{f}$ is equal to zero almost everywhere. The Fourier transform of f , denoted by $\hat{f}$, is defined by

$$\hat{f}(\xi) = \lim_{R \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^R e^{-ix\xi} f(x) dx,$$

where the convergence is understood to occur in $L^2(\mathbb{R})$. Similarly, the subspace $H_-^2(\mathbb{R})$ of $L^2(\mathbb{R})$ is defined by

$$H_-^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq (-\infty, 0]\}.$$

It can be proved that $H_+^2(\mathbb{R})$ and $H_-^2(\mathbb{R})$ are closed subspaces of $L^2(\mathbb{R})$. The spaces $H_+^2(\mathbb{R})$ and $H_-^2(\mathbb{R})$ are known as the Hardy space and the conjugate Hardy space, respectively.

Let $U(H_{\pm}^2(\mathbb{R}))$ be the set of all unitary operators on $H_{\pm}^2(\mathbb{R})$. It is a group with respect to the composition of mappings. Then the irreducible and unitary representations of U on $H_{\pm}^2(\mathbb{R})$ are the mappings $\pi_{\pm} : U \rightarrow U(H_{\pm}^2(\mathbb{R}))$ defined by

$$(\pi_{\pm}(b, a)f)(x) = \frac{1}{\sqrt{a}}f\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R},$$

for all points (b, a) in U and all functions $f \in H_{\pm}^2(\mathbb{R})$. A detailed study of the affine group and its representations can be found in [5]. In fact, π_+ is a square integrable representation of U on $H_+^2(\mathbb{R})$, which means that there exists a function φ in $H_+^2(\mathbb{R})$ such that

$$\|\varphi\|_{L^2(\mathbb{R})} = 1$$

and

$$(2.1) \quad \int_0^\infty \int_{-\infty}^\infty |(\varphi, \pi_+(b, a)\varphi)_{L^2(\mathbb{R})}|^2 \frac{db da}{a^2} < \infty.$$

a function $\varphi \in H_+^2(\mathbb{R})$ for which (2.1) is valid is called an admissible wavelet for the square integrable representation π_+ of U on $H_+^2(\mathbb{R})$.

THEOREM 2.2. *Let $v_{\pm} = \pi_{\pm} \circ c$. Then $v_{\pm}$ is the only infinite-dimensional, irreducible and unitary representation of $\mathbb{D}$ on $H_{\pm}^2(\mathbb{R})$. More explicitly, $v_{\pm} : \mathbb{D} \rightarrow U(H_{\pm}^2(\mathbb{R}))$ is given by*

$$(v_{\pm}(x, y)\varphi)(t) = \Delta(x, y)^{1/2} \varphi\left(t\Delta(x, y) - \frac{2x}{1-x^2-y^2}\right),$$

for all $\varphi \in H_{\pm}^2(\mathbb{R})$.

Proof Since the Cayley transform $c : \mathbb{D} \rightarrow U$ is a group homomorphism, it follows from the definition that $v_{\pm} : \mathbb{D} \rightarrow U(H_{\pm}^2(\mathbb{R}))$ is a group homomorphism. Let M be a nonzero and closed subspace of $H_{\pm}^2(\mathbb{R})$ invariant under $v_{\pm}$. Then for all $(x, y) \in \mathbb{D}$ and all $\varphi \in M$,

$$v_{\pm}(x, y)\varphi \in M,$$

which is the same as

$$(\pi_{\pm} \circ c)(x, y)\varphi \in M.$$

Since the Cayley transform is a biholomorphism, we have

$$\pi_{\pm}(b, a)\varphi \in M,$$

for all $(b, a) \in U$ and $\varphi \in H_{\pm}^2(\mathbb{R})$. But the irreducibility of $\pi_{\pm}$ of U in $H_{\pm}^2(\mathbb{R})$ implies that $M = H_{\pm}^2(\mathbb{R})$. Hence $v_{\pm}$ is an irreducible and unitary representation of $\mathbb{D}$ in $H_{\pm}^2(\mathbb{R})$. Let Γ be an irreducible and unitary representation of $\mathbb{D}$ on an infinite-dimensional, separable and complex Hilbert space X . Then $\Gamma \circ c^{-1} : U \rightarrow U(X)$ is an irreducible and unitary representation of U on X . So, $\Gamma \circ c^{-1}$ is either equivalent to either π_+ or π_- , which means that there is a unitary operator $U^{\pm} : X \rightarrow H_{\pm}^2(\mathbb{R})$ such that

$$U^{\pm} \left((\Gamma \circ c^{-1})(b, a) \right) = (\pi_{\pm}(b, a))U^{\pm}, \quad (b, a) \in U.$$

So,

$$U^\pm \Gamma(x, y) = v_\pm(x, y) U^\pm,$$

for all $(x, y) \in \mathbb{D}$. □

We use in this paper another irreducible and unitary representation $\rho_\pm$ of the affine group U on $L^2(\mathbb{R}_\pm)$, which is equivalent to $v_\pm$, which can be found in [14]. To make the paper self-contained, we first recall the definition.

Let $\rho_\pm : U \rightarrow U(L^2(\mathbb{R}_\pm))$ be the only infinitely-dimensional, irreducible and unitary representation of U on $L^2(\mathbb{R}_\pm)$ defined by

$$(\rho_\pm(b, a)u)(s) = a^{1/2} e^{-ibs} u(as), \quad s \in \mathbb{R}_\pm,$$

for all $u \in L^2(\mathbb{R}_\pm)$.

The next theorem gives the only infinite-dimensional, irreducible and unitary representations of $\mathbb{D}$ on $L^2(\mathbb{R})$, up to equivalence.

THEOREM 2.3. *Let $\mu_\pm = \rho_\pm \circ c$. Then $\mu_\pm$ is the only infinite-dimensional, irreducible and unitary representations, up to equivalence, of $\mathbb{D}$ on $L^2(\mathbb{R}_\pm)$. More explicitly, $\mu_\pm : \mathbb{D} \rightarrow U(L^2(\mathbb{R}_\pm))$ is given by*

$$(\mu_\pm(x, y)\varphi)(s) = \Delta(x, y)^{-1/2} e^{-i\pi xs} / (x^2 + (1 - y)^2) \varphi(\Delta(x, y)^{-1}s),$$

for all $\varphi \in L^2(\mathbb{R}_\pm)$ and $s \in \mathbb{R}_\pm$.

Next we recall the Duflo–Moore operator $D_\pm$ [9], which is an unbounded linear operator on $L^2(\mathbb{R}_\pm)$ and defined by

$$(D_\pm \varphi)(s) = |s|^{1/2} \varphi(s), \quad s \in \mathbb{R}_\pm.$$

Then it follows immediately that

$$D_\pm \mu_\pm(x, y) = \Delta(x, y)^{1/2} \mu_\pm(x, y) D_\pm, \quad (x, y) \in \mathbb{D}.$$

Let $f \in L^1(\mathbb{D}) \cap L^2(\mathbb{D})$. Then we define the Fourier transform $\hat{f}$ of f on $\hat{\mathbb{D}} = \{\mu_+, \mu_-\}$ by

$$(\hat{f}(\mu_\pm)\varphi)(s) = \int_{\mathbb{D}} f(x, y) (\mu_\pm(x, y) D_\pm \varphi)(s) \frac{dx dy}{(1 - x^2 - y^2)^2}, \quad s \in \mathbb{R}_\pm,$$

for all $\varphi \in L^2(\mathbb{R}_\pm)$. We have the following proposition from [13],

PROPOSITION 2.4. *Let $f \in L^1(\mathbb{D}) \cap L^2(\mathbb{D})$. Then $\hat{f}(\mu_+)$ and $\hat{f}(\mu_-)$ are integral operators on $L^2(\mathbb{R}_\pm)$ with kernels*

$$K_+^f(s, t) = \frac{1}{4} \frac{\sqrt{s}}{t} (2\pi)^{1/2} \left(\mathcal{F}_1(f \circ c^{-1}) \right) \left(\frac{\pi}{2} s, \frac{t}{s} \right), \quad s, t > 0,$$

and

$$K_-^f(s, t) = \frac{1}{4} \frac{\sqrt{|s|}}{|t|} (2\pi)^{1/2} \left(\mathcal{F}_1(f \circ c^{-1}) \right) \left(\frac{\pi}{2} s, \frac{t}{s} \right), \quad s, t < 0,$$

where $\mathcal{F}_1$ denotes the Euclidean Fourier transform of $f \circ c^{-1}$ with respect to the first variable.

We define the function K^f on $-\infty < s, t < \infty$ by

$$K^f(s, t) = \begin{cases} K_+^f(s, t), & s, t > 0, \\ K_-^f(s, t), & s, t < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $S_2^\pm$ be the space of all Hilbert–Schmidt operators on $L^2(\mathbb{R}_\pm)$. The norm in $S_2^\pm$ is denoted by $\|\cdot\|_{S_2^\pm}$. Let

$$S_2 = S_2^+ \cup S_2^-.$$

Let $L^2(\hat{\mathbb{D}}, S_2)$ be the space of all functions $F : \hat{\mathbb{D}} \rightarrow S_2$ such that $F(\mu_\pm) \in S_2^\pm$. It is easy to see that $L^2(\hat{\mathbb{D}}, S_2)$ is a Hilbert space with inner product $(\cdot, \cdot)_{L^2(\hat{\mathbb{D}}, S_2)}$ given by

$$(F, G)_{L^2(\hat{\mathbb{D}}, S_2)} = \text{tr}(F(\mu_+)G(\mu_+)^*) + \text{tr}(F(\mu_-)G(\mu_-)^*),$$

and the norm $\|\cdot\|_{L^2(\hat{\mathbb{D}}, S_2)}$ is given by

$$\|F\|_{L^2(\hat{\mathbb{D}}, S_2)}^2 = \|F(\mu_+)\|_{S_2^+}^2 + \|F(\mu_-)\|_{S_2^-}^2,$$

for all $F, G \in L^2(\hat{\mathbb{D}}, S_2)$. Then the Plancherel formula is given by

$$\|\hat{F}\|_{L^2(\mathbb{D}, S_2)}^2 = \|F\|_{L^2(\hat{\mathbb{D}})}^2,$$

and the Fourier inversion formula is given by

$$F(x, y) = \Delta(x, y)^{-1/2} (\text{tr}(D_+ \hat{F}(\mu) \mu_+(x, y)^*) + \text{tr}(D_- \hat{f}(\mu) \mu_-(x, y)^*)), \quad (x, y) \in \mathbb{D},$$

for all $F \in L^2(\mathbb{D})$.

We end this section with a connection between the Fourier transform and the Weyl transform. Let σ be a suitable measurable function on $\mathbb{R} \times \mathbb{R}$. Then the Weyl transform corresponding to the symbol σ is defined by

$$(W_\sigma f, g)_{L^2(\mathbb{R})} = (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi,$$

for all $f, g \in L^2(\mathbb{R})$. Here, $W(f, g)$ is the Wigner transform of f and g given by

$$W(f, g)(x, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\xi p} f\left(x + \frac{p}{2}\right) g\left(x - \frac{p}{2}\right) dp, \quad x, \xi \in \mathbb{R}.$$

Then we recall the following corollary from [13], which ensures that the Fourier transform on the unit disk is in fact a Weyl transform.

COROLLARY 2.5. *Let $f \in L^2(\mathbb{D})$. Then*

$$\hat{f}(\mu) = W_{\sigma_f},$$

where

$$\sigma_f = (2\pi)^{1/2} \mathcal{F}_2 T K^f$$

and $T : L^2(\mathbb{R} \times \mathbb{R}) \rightarrow L^2(\mathbb{R} \times \mathbb{R})$ is the unitary operator defined by

$$(Tg)(s, t) = g\left(s + \frac{t}{2}, s - \frac{t}{2}\right), \quad (s, t) \in \mathbb{R} \times \mathbb{R},$$

for all measurable functions $g \in L^2(\mathbb{R} \times \mathbb{R})$ and $\mathcal{F}_2$ is the Euclidean Fourier transform with respect to the second variable.

3. SQUARE-INTEGRABILITY

To be specific, we choose to look at the representation $\nu_+ : \mathbb{D} \rightarrow U(H_+^2(\mathbb{R}_+))$. We show in this section that the representation ν_+ is square-integrable in the sense that there exists a nonzero function φ in $H_+^2(\mathbb{R})$ with

$$\|\varphi\|_{L^2(\mathbb{R})} = 1$$

such that

$$(3.1) \quad \int_{\mathbb{D}} \left| (\varphi, \nu_+(x, y)\varphi)_{L^2(\mathbb{R})} \right|^2 d\mu(x, y) < \infty, \quad (x, y) \in \mathbb{D}.$$

Here φ is called an admissible wavelet and it satisfies the admissibility condition (3.1).

In the following result we give a characterization of all admissible wavelets.

THEOREM 3.1. *For all functions φ and ψ in $H_+^2(\mathbb{R})$,*

$$\int_{\mathbb{D}} |(\varphi, \nu_+(x, y)\psi)_{L^2(\mathbb{R})}|^2 d\mu(x, y) = 2\pi \int_{-\infty}^{\infty} |\widehat{\varphi}(\xi)|^2 d\xi \int_0^{\infty} |\widehat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|}.$$

Proof For all φ and ψ in $H_+^2(\mathbb{R})$, we get

$$\begin{aligned} & \int_{\mathbb{D}} \left| (\varphi, \nu_+(x, y)\psi)_{L^2(\mathbb{R})} \right|^2 \frac{dxdy}{(1-x^2-y^2)^2} \\ &= \int_{\mathbb{D}} \left| \int_{\mathbb{R}} \varphi(t) \overline{(\nu_+(x, y)\psi)(t)} dt \right|^2 \frac{dxdy}{(1-x^2-y^2)^2} \\ &= \int_{\mathbb{D}} \left| \int_{\mathbb{R}} \varphi(t) \overline{(\pi_+ \circ c(x, y)\psi)(t)} dt \right|^2 \frac{dxdy}{(1-x^2-y^2)^2} \\ &= \int_U \left| \int_{\mathbb{R}} \varphi(t) \overline{(\pi_+(b, a)\psi)(t)} dt \right|^2 \frac{dbda}{4a^2} \\ &= \int_U \left| \int_{\mathbb{R}} \varphi(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt \right|^2 \frac{dbda}{4a^3} \\ &= \int_U \left| \int_{\mathbb{R}} e^{ib\xi} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(a\xi)} d\xi \right|^2 \frac{dbda}{4a} \\ &= 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} |\mathcal{F}^{-1}(\widehat{\varphi}(\overline{D_a \widehat{\psi}}))(b)|^2 \frac{dbda}{4a} \\ &= 2\pi \int_0^{\infty} \int_{-\infty}^{\infty} |\widehat{\varphi}(\xi) \overline{(D_a \widehat{\psi})(\xi)}|^2 \frac{d\xi da}{4a} \end{aligned}$$

$$= 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} |\widehat{\varphi}(\xi)|^2 |\widehat{\psi}(a\xi)|^2 \frac{d\xi da}{4a}.$$

So,

$$\int_{\mathbb{D}} |(\varphi, \mu_+(x, y)\psi)_{L^2(\mathbb{R})}|^2 \frac{dx dy}{(1-x^2-y^2)^2} = 2\pi \int_{-\infty}^{\infty} |\widehat{\varphi}(\xi)|^2 \left(\int_0^{\infty} |\widehat{\psi}(a\xi)|^2 \frac{da}{4a} \right) d\xi.$$

A simple calculation shows that

$$\int_0^{\infty} |\widehat{\psi}(a\xi)|^2 \frac{da}{4a} = \int_0^{\infty} |\widehat{\psi}(\eta)|^2 \frac{d\eta}{4|\eta|}.$$

This implies that

$$\int_{\mathbb{D}} |(\varphi, \nu_+(x, y)\psi)_{L^2(\mathbb{R})}|^2 \frac{dx dy}{(1-x^2-y^2)^2} = 2\pi \left(\int_{-\infty}^{\infty} |\widehat{\varphi}(\xi)|^2 d\xi \right) \left(\int_0^{\infty} |\widehat{\psi}(\eta)|^2 \frac{d\eta}{|\eta|} \right).$$

□

COROLLARY 3.2. *The set AW of all admissible wavelets for the square-integrable representation $\nu_+ : \mathbb{D} \rightarrow U(H_+^2(\mathbb{R}_+))$ is given by*

$$AW = \left\{ \varphi \in H_{\pm}^2(\mathbb{R}) : \|\varphi\|_{L^2(\mathbb{R})} = 1, \int_0^{\infty} |\widehat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty \right\},$$

and for all $\varphi \in AW$, the wavelet constant c_{φ} is given by

$$c_{\varphi} = 2\pi \int_0^{\infty} |\widehat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|}.$$

4. LOCALIZATION OPERATORS

In this section we will first state the following result on the Schatten-von Neumann property of the localization operators on the unit disk. This is special case of the corresponding result for localization operators on affine group in the paper by Wong [20].

THEOREM 4.1. *Let φ and c_{φ} be as in the preceding corollary. Let $F \in L^p(\mathbb{D})$, $1 \leq p \leq \infty$. Then the localization operators $L_{F,\varphi} : H_+^2(\mathbb{R}) \rightarrow H_+^2(\mathbb{R})$ associated to the admissible wavelet φ and the symbol F defined by*

$$(L_{F,\varphi} f, g)_{L^2(\mathbb{R})} = \frac{1}{c_{\varphi}} \int_0^{\infty} \int_{-\infty}^{\infty} F(x, y) (f, \mu_+(x, y)\varphi)_{L^2(\mathbb{R})} (\mu_+(x, y)\varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1-x^2-y^2)^2},$$

for all f and g in $H_+^2(\mathbb{R})$ is in the Schatten- von Neumann class S_p and

$$\|L_{F,\varphi}\|_{S_p} \leq \left(\frac{1}{c_{\varphi}} \right)^{1/p} \|F\|_{L^p(\mathbb{D})}.$$

In the next sections we give the explicit formulas for the localization operators $L_{F,\varphi} : H_+^2(\mathbb{R}) \rightarrow H_+^2(\mathbb{R})$ under different conditions imposed on the function F . These results are the extensions of results obtained in [20] for the affine group U .

5. PARACOMMUTATORS

Let F be a function on $\mathbb{D}$ given by

$$F(x, y) = \alpha\left(\frac{1 - x^2 - y^2}{x^2 + (1 - y)^2}\right) \beta\left(\frac{2x}{x^2 + (1 - y)^2}\right), \quad (x, y) \in \mathbb{D},$$

where α and β are appropriate functions on $\mathbb{R}$ and $(0, \infty)$, respectively. Now, using the definition of the localization operator associated to the admissible wavelet φ and the symbol F , we get

$$\begin{aligned} & (L_{F, \varphi} f, g)_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_{\mathbb{D}} F(x, y) (f, \mu_+(x, y) \varphi)_{L^2(\mathbb{R})} (\mu_+(x, y) \varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1 - x^2 - y^2)^2} \\ &= \frac{1}{c_\varphi} \int_{\mathbb{D}} \alpha\left(\frac{1 - x^2 - y^2}{x^2 + (1 - y)^2}\right) \beta\left(\frac{2x}{x^2 + (1 - y)^2}\right) \\ & \quad (f, \mu_+(x, y) \varphi)_{L^2(\mathbb{R})} (\mu_+(x, y) \varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1 - x^2 - y^2)^2}, \end{aligned}$$

for all f and g in $H_+^2(\mathbb{R})$. Letting $\frac{1 - x^2 - y^2}{x^2 + (1 - y)^2} = a$ and $\frac{2x}{x^2 + (1 - y)^2} = b$, and using the Cayley transform

$$c(x, y) = \left(\frac{2x}{x^2 + (1 - y)^2}, \frac{1 - x^2 - y^2}{x^2 + (1 - y)^2} \right) \Rightarrow c^{-1}(b, a) = (x, y),$$

we get

$$\frac{db da}{4a^2} = \frac{dx dy}{(1 - x^2 - y^2)^2},$$

and

$$\begin{aligned} & (L_{F, \varphi} f, g)_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_U \alpha(a) \beta(b) (f, \mu_+ \circ c^{-1}(b, a) \varphi)_{L^2(\mathbb{R})} (\mu_+ \circ c^{-1}(b, a) \varphi, g)_{L^2(\mathbb{R})} \frac{db da}{4a^2} \\ &= \frac{1}{c_\varphi} \int_U \alpha(a) \beta(b) \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{a}} \overline{\varphi\left(\frac{t - b}{a}\right)} dt \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \varphi\left(\frac{s - b}{a}\right) \overline{g(s)} ds \frac{db da}{4a^2} \\ &= \frac{1}{c_\varphi} \int_U \alpha(a) \beta(b) \int_{-\infty}^{\infty} \widehat{f}(\xi) \sqrt{a} e^{ib\xi} \overline{\widehat{\varphi}(a\xi)} d\xi \int_{-\infty}^{\infty} \sqrt{a} e^{-ib\eta} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta \frac{db da}{4a^2} \\ &= \frac{1}{c_\varphi} \int_0^\infty \int_{-\infty}^\infty \alpha(a) \beta(b) \int_{-\infty}^\infty \widehat{f}(\xi) e^{ib\xi} \overline{\widehat{\varphi}(a\xi)} d\xi \int_{-\infty}^\infty e^{-ib\eta} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta \frac{db da}{4a} \\ &= \frac{1}{c_\varphi} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta \frac{da}{4a} d\xi \int_{-\infty}^\infty \beta(b) e^{-ib(\eta - \xi)} db \\ &= \frac{\sqrt{2\pi}}{c_\varphi} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \alpha(a) \widehat{\beta}(\eta - \xi) \widehat{f}(\xi) \widehat{\psi}_a(\xi) \widehat{\varphi}_a(\eta) \overline{\widehat{g}(\eta)} d\eta \frac{da}{4a} d\xi \\ &= \frac{\sqrt{2\pi}}{c_\varphi} \int_{-\infty}^\infty \int_{-\infty}^\infty \left(\int_0^\infty \alpha(a) \widehat{\psi}_a(\xi) \widehat{\varphi}_a(\eta) \frac{da}{4a} \right) \widehat{\beta}(\eta - \xi) \widehat{f}(\xi) \overline{\widehat{g}(\eta)} d\eta d\xi \\ &= \frac{\sqrt{2\pi}}{c_\varphi} \int_{-\infty}^\infty \int_{-\infty}^\infty A(\xi, \eta) \widehat{\beta}(\eta - \xi) \widehat{f}(\xi) \overline{\widehat{g}(\eta)} d\eta d\xi, \end{aligned}$$

where

$$A(\xi, \eta) = \int_0^\infty \alpha(a) \widehat{\psi}_a(\xi) \widehat{\varphi}_a(\eta) \frac{da}{4a}, \varphi_a(x) = \frac{1}{a} \varphi\left(\frac{x}{a}\right), \psi(x) = \overline{\varphi(-x)}.$$

Thus, the localization operator $L_{F,\varphi}$ is a paracommutator with kernel A and symbol β .

6. PARAPRODUCT CONNECTION

In this section we consider the case when the function α in the symbol F is a constant, that is,

$$F(x, y) = \beta \left(\frac{2x}{x^2 + (1-y)^2} \right) \alpha \left(\frac{1-x^2-y^2}{x^2 + (1-y)^2} \right)$$

and

$$\alpha \left(\frac{1-x^2-y^2}{x^2 + (1-y)^2} \right) = k$$

for all $(x, y) \in \mathbb{D}$, where $k \neq 0$ is a constant. The localization operator $L_{F,\varphi}$ associated to the admissible wavelet φ and this symbol F is given by

$$\begin{aligned} & (L_{F,\varphi} f, g)_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_{\mathbb{D}} F(x, y) (f, \mu_+(x, y) \varphi)_{L^2(\mathbb{R})} (\mu_+(x, y) \varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1-x^2-y^2)^2} \\ &= \frac{1}{c_\varphi} \int_{\mathbb{D}} \beta \left(\frac{2x}{x^2 + (1-y)^2} \right) \alpha \left(\frac{1-x^2-y^2}{x^2 + (1-y)^2} \right) \\ & \quad (f, \mu_+(x, y) \varphi)_{L^2(\mathbb{R})} (\mu_+(x, y) \varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1-x^2-y^2)^2}. \end{aligned}$$

Letting $\frac{1-x^2-y^2}{x^2+(1-y)^2} = a$ and $\frac{2x}{x^2+(1-y)^2} = b$, and using the Cayley transform

$$c(x, y) = \left(\frac{2x}{x^2 + (1-y)^2}, \frac{1-x^2-y^2}{x^2 + (1-y)^2} \right) \Rightarrow c^{-1}(b, a) = (x, y),$$

we get

$$\frac{db da}{4a^2} = \frac{dx dy}{(1-x^2-y^2)^2}$$

and

$$\begin{aligned} & (L_{F,\varphi} f, g)_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_U \alpha(a) \beta(b) (f, \mu_+ \circ c^{-1}(b, a) \varphi)_{L^2(\mathbb{R})} (\mu_+ \circ c^{-1}(b, a) \varphi, g)_{L^2(\mathbb{R})} \frac{db da}{4a^2} \\ &= \frac{k}{c_\varphi} \int_U \beta(b) (f, \mu_+ \circ c^{-1}(b, a) \varphi)_{L^2(\mathbb{R})} (\mu_+ \circ c^{-1}(b, a) \varphi, g)_{L^2(\mathbb{R})} \frac{db da}{4a^2} \\ &= \frac{k\sqrt{2\pi}}{c_\varphi} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty \widehat{\psi}_a(\xi) \widehat{\varphi}_a(\eta) \widehat{\beta}(\eta - \xi) \widehat{f}(\xi) \overline{\widehat{g}(\eta)} \frac{da}{4a} d\xi d\eta, \end{aligned}$$

where

$$\varphi_a(x) = \frac{1}{a} \varphi\left(\frac{x}{a}\right), \psi(x) = \overline{\varphi(-x)},$$

for all $x \in \mathbb{R}$. Next, we define

$$p(x) = \int_0^\infty (\beta(\psi_a * f) * \varphi_a)(x) \frac{da}{4a}, \quad x \in \mathbb{R}.$$

Then

$$\begin{aligned} \widehat{p}(\eta) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty (\beta(\psi_a * f) * \varphi_a)(x) e^{-ix\eta} \frac{da}{4a} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \beta(\psi_a * f)(y) \varphi_a(x-y) dy e^{-ix\eta} \frac{da}{4a} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \beta(\psi_a * f)(y) \varphi_a(x-y) e^{-i(x-y)\eta} e^{-iy\eta} dy \frac{da}{4a} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{-\infty}^\infty \beta(\psi_a * f)(y) e^{-iy\eta} dy \int_{-\infty}^\infty \varphi_a(x-y) e^{-i(x-y)\eta} dx \frac{da}{4a} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{2\pi} (\beta(\psi_a * f))^\wedge(\eta) \sqrt{2\pi} \widehat{\varphi_a}(\eta) \frac{da}{4a} \\ &= \sqrt{2\pi} \int_0^\infty (\widehat{\beta * (\psi_a * f)})^\wedge(\eta) \widehat{\varphi_a}(\eta) \frac{da}{4a} \\ &= \sqrt{2\pi} \int_0^\infty \int_{-\infty}^\infty \widehat{\beta}(\eta - \xi) (\widehat{\psi_a * f})(\xi) \widehat{\varphi_a}(\eta) \frac{da}{4a} d\xi \\ &= 2\pi \int_0^\infty \int_{-\infty}^\infty \widehat{\beta}(\eta - \xi) \widehat{\psi_a}(\xi) \widehat{f}(\xi) \widehat{\varphi_a}(\eta) \frac{da}{4a} d\xi, \quad \eta \in \mathbb{R}. \end{aligned}$$

From this we get

$$(L_{F,\varphi} f)(x) = \frac{k}{c_\varphi \sqrt{2\pi}} \int_0^\infty (\beta(\psi_a * f) * \varphi_a)(x) \frac{da}{4a}, \quad x \in \mathbb{R}.$$

Then we have

$$\begin{aligned} (L_{F,\varphi} f, g)_{L^2(\mathbb{R})} &= \frac{k}{c_\varphi \sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty (\beta(\psi_a * f) * \varphi_a)(x) \frac{da}{4a} \overline{g(x)} dx \\ &= \frac{k}{c_\varphi \sqrt{2\pi}} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty \beta(\psi_a * f)(y) \varphi_a(x-y) \frac{da}{4a} \overline{g(x)} dx dy \\ &= \frac{k}{c_\varphi \sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty \beta(\psi_a * f)(y) \int_{-\infty}^\infty \overline{\psi_a(y-x) g(x)} dx dy \frac{da}{4a} \\ &= \frac{k}{c_\varphi \sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty \beta(\psi_a * f)(y) \overline{(\psi_a * g)(y)} dy \frac{da}{4a} \\ &= \frac{k}{c_\varphi \sqrt{2\pi}} \int_{-\infty}^\infty \int_0^\infty (\psi_a * f)(y) \overline{(\psi_a * g)(y)} \frac{da}{4a} \beta(y) dy \\ &= \frac{k}{c_\varphi \sqrt{2\pi}} \int_{-\infty}^\infty M_\psi(f, g)(y) \beta(y) dy, \end{aligned}$$

where

$$M_\psi(f, g)(y) = \int_0^\infty (\psi_a * f)(y) \overline{(\psi_a * g)(y)} \frac{da}{4a}, \quad y \in \mathbb{R},$$

is the paraproduct of f and g with respect to the admissible wavelet ψ and

$$\psi(x) = \overline{\varphi(-x)}, \psi_a(x) = \frac{1}{a} \psi\left(\frac{x}{a}\right),$$

and for all $x \in \mathbb{R}$.

7. FOURIER MULTIPLIERS

Let F be a function on $\mathbb{D}$ given by

$$F(x, y) = \alpha\left(\frac{1 - x^2 - y^2}{x^2 + (1 - y)^2}\right) \beta\left(\frac{2x}{x^2 + (1 - y)^2}\right)$$

with $\beta\left(\frac{2x}{x^2 + (1 - y)^2}\right) = k$, where k is a nonzero constant. Then for all f and g in $H_+^2(\mathbb{R})$,

$$\begin{aligned} & (L_{F, \varphi} f, g)_{L^2(\mathbb{R})} \\ &= \frac{1}{c_\varphi} \int_D F(x, y) (f, \mu_+(x, y) \varphi)_{L^2(\mathbb{R})} (\mu_+(x, y) \varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1 - x^2 - y^2)^2} \\ &= \frac{1}{c_\varphi} \int_D \alpha\left(\frac{1 - x^2 - y^2}{x^2 + (1 - y)^2}\right) \beta\left(\frac{2x}{x^2 + (1 - y)^2}\right) \\ & \quad (f, \mu_+(x, y) \varphi)_{L^2(\mathbb{R})} (\mu_+(x, y) \varphi, g)_{L^2(\mathbb{R})} \frac{dx dy}{(1 - x^2 - y^2)^2}. \end{aligned}$$

Using the Cayley transform $c(x, y) = \left(\frac{2x}{x^2 + (1 - y)^2}, \frac{1 - x^2 - y^2}{x^2 + (1 - y)^2}\right)$, we have $c^{-1}(b, a) = (x, y)$. As in the earlier cases, using $\frac{1 - x^2 - y^2}{x^2 + (1 - y)^2} = a$ and $\frac{2x}{x^2 + (1 - y)^2} = b$ we get

$$\frac{db da}{4a^2} = \frac{dx dy}{(1 - x^2 - y^2)^2}$$

and

$$\begin{aligned} & (L_{F, \varphi} f, g)_{L^2(\mathbb{R})} \\ &= \frac{k}{c_\varphi} \int_U \alpha(a) (f, \mu_+ \circ c^{-1}(b, a) \varphi)_{L^2(\mathbb{R})} (\mu_+ \circ c^{-1}(b, a) \varphi, g)_{L^2(\mathbb{R})} \frac{db da}{4a^2} \\ &= \frac{k}{c_\varphi} \int_U \alpha(a) \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{a}} \varphi\left(\frac{t - b}{a}\right) dt \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \varphi\left(\frac{s - b}{a}\right) \overline{g(s)} ds \frac{db da}{4a^2} \\ &= \frac{k}{c_\varphi} \int_U \alpha(a) \int_{-\infty}^{\infty} \widehat{f}(\xi) \sqrt{a} e^{ib\xi} \overline{\widehat{\varphi}(a\xi)} d\xi \int_{-\infty}^{\infty} \sqrt{a} e^{-ib\eta} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta \frac{db da}{4a^2} \\ &= \frac{k}{c_\varphi} \int_0^\infty \int_{-\infty}^\infty \alpha(a) \int_{-\infty}^\infty \widehat{f}(\xi) e^{ib\xi} \overline{\widehat{\varphi}(a\xi)} d\xi \int_{-\infty}^\infty e^{-ib\eta} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta \frac{db da}{4a} \\ &= \frac{k}{c_\varphi} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} e^{-ib(\eta - \xi)} db d\eta d\xi \frac{da}{4a}. \end{aligned}$$

Let $h(x) = e^{-x^2/2}$, $x \in \mathbb{R}$. Then for all $\varepsilon > 0$, we define I_ε by

$$I_\varepsilon = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-b^2 \varepsilon^2 / 2} \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} e^{-ib(\eta - \xi)} db d\eta d\xi \frac{da}{4a}.$$

Then we have

$$\begin{aligned} I_\varepsilon &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \left(\frac{1}{\varepsilon} \int_{-\infty}^\infty h(b\varepsilon) e^{ib\varepsilon(\frac{\xi - \eta}{\varepsilon})} \varepsilon db \right) \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta d\xi \frac{da}{4a} \\ &= \sqrt{2\pi} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\varepsilon} h\left(\frac{\xi - \eta}{\varepsilon}\right) \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta d\xi \frac{da}{4a} \\ &= \sqrt{2\pi} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty h_\varepsilon(\xi - \eta) \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta d\xi \frac{da}{4a} \end{aligned}$$

$$\begin{aligned}
&= \sqrt{2\pi} \int_0^\infty \int_{-\infty}^\infty \left(\int_{-\infty}^\infty h_\varepsilon(\xi - \eta) \widehat{\varphi}(a\eta) \overline{\widehat{g}(\eta)} d\eta \right) \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} d\xi \frac{da}{4a} \\
&= \sqrt{2\pi} \int_0^\infty \int_{-\infty}^\infty \left((\widehat{\varphi_a \widehat{g}}) * h_\varepsilon \right) (\xi) \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} d\xi \frac{da}{4a},
\end{aligned}$$

where h_ε is the Friedrich mollifier of h . Now there exists a sequence $\{\varepsilon_j\}_{j=1}^\infty$ of positive numbers such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ and

$$\widehat{\varphi_a \widehat{g}} * h_{\varepsilon_j} \rightarrow \sqrt{2\pi} \widehat{\varphi_a \widehat{g}}$$

in $L^2(\mathbb{R})$. Thus,

$$I_{\varepsilon_j} \rightarrow 2\pi \int_0^\infty \int_{-\infty}^\infty \widehat{\varphi_a}(\xi) \overline{\widehat{g}(\xi)} \alpha(a) \widehat{f}(\xi) \overline{\widehat{\varphi}(a\xi)} \frac{da}{4a} d\xi = 2\pi \int_0^\infty \int_{-\infty}^\infty \widehat{g}(\xi) \alpha(a) \widehat{f}(\xi) |\widehat{\varphi}(a\xi)|^2 d\xi \frac{da}{4a}.$$

Using Lebesgue's dominated convergence theorem, we see that

$$I_{\varepsilon_j} \rightarrow \frac{c_\varphi}{k} (L_{F,\varphi} f, g)_{L^2(\mathbb{R})},$$

as $j \rightarrow \infty$. Therefore

$$\begin{aligned}
(L_{F,\varphi} f, g)_{L^2(\mathbb{R})} &= \frac{2\pi k}{c_\varphi} \int_{-\infty}^\infty \int_0^\infty \alpha(a) |\widehat{\varphi}(a\xi)|^2 \frac{da}{4a} \widehat{g}(\xi) \widehat{f}(\xi) d\xi \\
&= \int_{-\infty}^\infty m(\xi) \widehat{g}(\xi) \widehat{f}(\xi) d\xi \\
&= \int_{-\infty}^\infty (\widehat{T_m f})(\xi) \widehat{g}(\xi) d\xi \\
&= \int_{-\infty}^\infty (T_m f)(x) \overline{g(x)} dx \\
&= (T_m f, g)_{L^2(\mathbb{R})},
\end{aligned}$$

where T_m is the Fourier multiplier with symbol m given by

$$m(\xi) = \frac{2\pi k}{c_\varphi} \int_0^\infty \alpha(a) |\widehat{\varphi}(a\xi)|^2 \frac{da}{4a}.$$

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