

Trace Class Ideals of H^* -Algebras of Pseudo-Differential Operators on Direct Products of Locally Compact, Hausdorff and Abelian Groups

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Abstract: We construct a H^* -algebra of pseudo-differential operators on a direct product of two locally compact, Hausdorff and abelian groups. A characterization of the trace class operators in the H^* -algebra is given.

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1 Fourier Analysis and Pseudo-Differential Operators on Direct Products of Locally Compact, Hausdorff and Abelian Groups

We begin with a class of topological groups. Let A and B be locally compact, Hausdorff and abelian groups with binary operations given, respectively, by $\circ$ and $\bullet$. Let G be the direct product of A and B given by

$$G = A \times B = \{(a, b) : a \in A, b \in B\}.$$

It is well known that G is a group with respect to the binary operation $\cdot$ given by

$$(a, b) \cdot (c, d) = (a \circ c, b \bullet d), \quad (a, b), (c, d) \in G.$$

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It is also easy to see that G is a locally compact, Hausdorff and abelian group with respect to $\cdot$. We denote by μ_A and μ_B , respectively, the Haar measure on A and the Haar measure on B . Let $C_0(A \times B)$ be the set of all continuous complex-valued functions on $A \times B$ with compact support. Let Λ be the linear functional on $C_0(A \times B)$ given by

$$\Lambda(f) = \int_A \int_B f(a, b) d\mu_B(b) d\mu_A(a)$$

for all $f \in C_0(A \times B)$ such that

$$f(A \times B) \subset [0, \infty) \Rightarrow \Lambda(f) \in [0, \infty).$$

By Theorem 2.14 in the book [12] by Rudin on real and complex analysis, there exists a unique Radon measure $\mu_{A \times B}$ on $A \times B$, which we also denote by $\mu_A \times \mu_B$, such that

$$\Lambda(f) = \int_A \int_B f(a, b) d\mu_B(b) d\mu_A(a) = \int_{A \times B} f(a, b) d\mu_{A \times B}(a, b)$$

for all $f \in C_0(A \times B)$. That $\mu_A \times \mu_B$ is invariant follows from the invariance of Λ , i.e., $\mu_A \times \mu_B$ is the Haar measure on $A \times B$. Then the Haar measure on G is given by μ , where μ is the product measure of μ_A and μ_B , namely,

$$\mu = \mu_A \times \mu_B.$$

We have the following theorem.

Theorem 1.1 *The dual group $\widehat{G}$ of G is isomorphic and homeomorphic to the direct product $\widehat{A} \times \widehat{B}$ of the dual group of A and the dual group of B .*

A proof of Theorem 1.1 can be found on page 139 of the book [6] by Loomis.

Let $f \in L^1(G)$. Then we define the Fourier transform $\mathcal{F}_G f$ of f to be the function on $\widehat{G}$ by

$$(\mathcal{F}_G f)(\chi) = \int_G \overline{\chi(g)} f(g) d\mu(g), \quad \chi \in \widehat{G}.$$

In fact, by Fubini's theorem,

$$\begin{aligned} (\mathcal{F}_G f)(\chi) &= \int_A \int_B \overline{\alpha(a)\beta(b)} f(a, b) d\mu_B(b) d\mu_A(a) \\ &= \int_A \overline{\alpha(a)} \left(\int_B \overline{\beta(b)} f(a, b) d\mu_B(b) \right) d\mu_A(a) \\ &= (\mathcal{F}_A \mathcal{F}_B f)(\alpha, \beta) \end{aligned}$$

for all $\chi = (\alpha, \beta) \in \widehat{G} = \widehat{A} \times \widehat{B}$. Therefore

$$\mathcal{F}_G f = \mathcal{F}_A \mathcal{F}_B f, \quad f \in L^1(G) = L^1(A \times B).$$

Similarly,

$$\mathcal{F}_G f = \mathcal{F}_B \mathcal{F}_A f, \quad f \in L^1(G) = L^1(A \times B).$$

The following theorem is the Plancherel formula for the Fourier transform $\mathcal{F}_G$ on G .

Theorem 1.2 *The Fourier transform, initially defined on $L^1(G) \cap L^2(G)$, can be extended uniquely to a surjective isometry $\mathcal{F}_G : L^2(G) \rightarrow L^2(\widehat{G})$, which is called the Fourier transform on $L^2(G)$. The Haar measure ν on $\widehat{G}$ is assumed to be normalized so that*

$$(\mathcal{F}_G f_1, \mathcal{F}_G f_2)_{L^2(\widehat{G})} = (f_1, f_2)_{L^2(G)}$$

for all f_1 and f_2 in $L^2(G)$.

Remark 1.3 The Haar measure ν on $\widehat{G}$ can be written as $\nu_A \times \nu_B$, where ν_A is the Haar measure on $\widehat{A}$ and ν_B is the Haar measure on $\widehat{B}$.

From Theorem 1.2, we obtain the Fourier inversion formula for $\mathcal{F}_G$.

Theorem 1.4 *For all $f \in L^2(G)$,*

$$f = \int_{\widehat{G}} \chi(\cdot) (\mathcal{F}_G f)(\chi) d\nu(\chi).$$

The Fourier inversion formula is equivalent to

$$f = \mathcal{F}_G^{-1} \mathcal{F}_G f, \quad f \in L^2(G),$$

where $\mathcal{F}_G^{-1} : L^2(\widehat{G}) \rightarrow L^2(G)$ is given by

$$(\mathcal{F}_G^{-1} \varphi)(g) = \int_{\widehat{G}} \chi(g) \varphi(\chi) d\nu(\chi), \quad g \in G,$$

for all $\varphi \in L^2(\widehat{G})$.

Remark 1.5 Since

$$\mathcal{F}_G = \mathcal{F}_A \mathcal{F}_B = \mathcal{F}_B \mathcal{F}_A,$$

it follows that

$$\mathcal{F}_G^{-1} = \mathcal{F}_B^{-1} \mathcal{F}_A^{-1} = \mathcal{F}_A^{-1} \mathcal{F}_B^{-1}.$$

The Fourier analysis hitherto described can be found in another book [13] by Rudin in Fourier analysis on groups.

There is a huge literature of pseudo-differential operators on a variety of groups. See, for instance, [3, 4, 5, 8, 9, 10, 14, 15, 18]. These groups are standard groups arising in various contexts. There are geometric configurations that can be realized as direct products of standard locally compact, Hausdorff and abelian groups, which can be studied as abelian groups instead of noncommutative groups. We construct in this paper a H^* -algebra of pseudo-differential operators on the direct product of two locally compact, Hausdorff and abelian groups.

2 Pseudo-Differential Operators on

$$L^2(G) = L^2(A \times B)$$

Let σ be a measurable function on

$$G \times \widehat{G} = (A \times B) \times (\widehat{A} \times \widehat{B}) = \{(a, b, \alpha, \beta) : (a, b) \in A \times B, (\alpha, \beta) \in \widehat{A} \times \widehat{B}\}.$$

Then we define the pseudo-differential operator T_σ on $G = A \times B$ by

$$(T_\sigma f)(g) = \int_{\widehat{G}} \chi(g) \sigma(g, \chi) (\mathcal{F}_G f)(\chi) d\nu(\chi), \quad g \in G,$$

for all suitable functions f on G such that the integral exists. Factoring G into A and B gives

$$(T_\sigma f)(a, b) = \int_{\widehat{A}} \int_{\widehat{B}} \alpha(a) \beta(b) \sigma(a, b, \alpha, \beta) (\mathcal{F}_A \mathcal{F}_B f)(\alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \quad (2.1)$$

for all $(a, b) \in A \times B$, provided that the integral exists.

Theorem 2.1 *Let $\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Then the pseudo-differential operator $T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B)$ is a bounded linear operator. Moreover,*

$$\|T_\sigma\|_{B(L^2(A \times B))} \leq \|\sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))},$$

where $\|\cdot\|_{B(L^2(A \times B))}$ is the norm in the C^* -algebra of all bounded linear operators on $L^2(A \times B)$.

Proof Let $f \in L^2(G)$. Then by Minkowski's inequality in integral form,

$$\begin{aligned} \|T_\sigma f\|_{L^2(G)} &= \left(\int_G |(T_\sigma f)(g)|^2 d\mu(g) \right)^{1/2} \\ &= \left(\int_G \left| \int_{\widehat{G}} \chi(g) \sigma(g, \chi) (\mathcal{F}_G f)(\chi) d\nu(\chi) \right|^2 d\mu(g) \right)^{1/2} \\ &\leq \int_{\widehat{G}} \left(\int_G |\sigma(g, \chi)|^2 |(\mathcal{F}_G f)(\chi)|^2 d\mu(g) \right)^{1/2} d\nu(\chi) \\ &= \int_{\widehat{G}} |(\mathcal{F}_G f)(\chi)| \left(\int_G |\sigma(g, \chi)|^2 d\mu(g) \right)^{1/2} d\nu(\chi). \end{aligned}$$

By the Schwarz inequality and the Plancherel formula,

$$\begin{aligned} \|T_\sigma f\|_{L^2(G)} &\leq \left(\int_{\widehat{G}} \int_G |\sigma(g, \chi)|^2 d\mu(g) d\nu(\chi) \right)^{1/2} \left(\int_{\widehat{G}} |(\mathcal{F}_G f)(\chi)|^2 d\nu(\chi) \right)^{1/2} \\ &= \left(\int_{\widehat{G}} \int_G |\sigma(g, \chi)|^2 d\mu(g) d\nu(\chi) \right)^{1/2} \|\mathcal{F}_G f\|_{L^2(\widehat{G})} \\ &= \left(\int_{\widehat{G}} \int_G |\sigma(g, \chi)|^2 d\mu(g) d\nu(\chi) \right)^{1/2} \|f\|_{L^2(G)}. \end{aligned}$$

Factoring G into A and B again, we obtain

$$\begin{aligned} &\|T_\sigma f\|_{L^2(A \times B)} \\ &\leq \left(\int_{\widehat{A}} \int_{\widehat{B}} \int_A \int_B |\sigma(a, b, \alpha, \beta)|^2 d\mu_B(b) d\mu_A(a) d\nu_B(\beta) d\nu_A(\alpha) \right)^{1/2} \|f\|_{L^2(G)} \\ &= \|\sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} \|f\|_{L^2(A \times B)}. \end{aligned}$$

□

Theorem 2.2 *A bounded linear operator T from $L^2(A \times B)$ into $L^2(A \times B)$ is a Hilbert–Schmidt operator if and only if*

$$T = T_\sigma$$

with

$$\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B})).$$

Moreover, the Hilbert–Schmidt norm $\|T_\sigma\|_{S_2(A \times B)}$ of the pseudo-differential operator T_σ from $L^2(A \times B)$ into $L^2(A \times B)$ is given by

$$\|T_\sigma\|_{S_2(A \times B)} = \|\sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}.$$

Proof Let $\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Then for all functions $f \in L^2(A \times B)$, it follows from (2.1) that

$$\begin{aligned} & (T_\sigma f)(a, b) \\ &= \int_{\widehat{A}} \int_{\widehat{B}} \alpha(a) \beta(b) \sigma(a, b, \alpha, \beta) \times \\ & \quad \times \left(\int_A \int_B \alpha(c^{-1}) \beta(d^{-1}) f(c, d) d\mu_B(d) d\mu_A(c) \right) d\nu_B(\beta) d\nu_A(\alpha) \\ &= \int_A \int_B \left(\int_{\widehat{A}} \int_{\widehat{B}} \alpha(a \circ c^{-1}) \beta(b \bullet d^{-1}) \sigma(a, b, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \right) \times \\ & \quad \times f(c, d) d\mu_B(d) d\mu_A(c) \\ &= \int_A \int_B K(a, b, c, d) f(c, d) d\mu_B(d) d\mu_A(c), \end{aligned}$$

where

$$\begin{aligned} & K(a, b, c, d) \\ &= \int_{\widehat{A}} \int_{\widehat{B}} \alpha(a \circ c^{-1}) \beta(b \bullet d^{-1}) \sigma(a, b, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \\ &= ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1}) \end{aligned} \tag{2.2}$$

for all (a, b) and (c, d) in $A \times B$, where $(\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma$ is the inverse Fourier transform of σ with respect to the second pair of coordinates. Therefore by

$$\|K\|_{L^2((A \times B) \times (A \times B))}^2$$

$$\begin{aligned}
&= \int_A \int_B \left(\int_A \int_B |K(a, b, c, d)|^2 d\mu_B(d) d\mu_A(c) \right) d\mu_B(b) d\mu_A(a) \\
&= \int_A \int_B \left(\int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1})|^2 d\mu_B(d) d\mu_A(c) \right) \times \\
&\quad \times d\mu_B(b) d\mu_A(a).
\end{aligned}$$

Let $x = a \circ c^{-1}$ and $y = b \bullet d^{-1}$. Then by Plancherel's formula,

$$\begin{aligned}
&\|K\|_{L^2((A \times B) \times (A \times B))}^2 \\
&= \int_A \int_B \left(\int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, x, y)|^2 d\mu_B(y) d\mu_A(x) \right) d\mu_B(b) d\mu_A(a) \\
&= \int_A \int_B \left(\int_{\hat{A}} \int_{\hat{B}} |\sigma(a, b, \alpha, \beta)|^2 d\nu_B(\beta) d\nu_A(\alpha) \right) d\mu_B(b) d\mu_A(a) \\
&= \|\sigma\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))}^2.
\end{aligned}$$

Let $T : L^2(A \times B) \rightarrow L^2(A \times B)$ be a Hilbert–Schmidt operator. Then there exists a function $K \in L^2((A \times B) \times (A \times B))$ such that for all $f \in L^2(A \times B)$,

$$(Tf)(a, b) = \int_A \int_B K(a, b, c, d) f(c, d) d\mu_B(d) d\mu_A(c), \quad (a, b) \in A \times B.$$

By Plancherel's formula,

$$\begin{aligned}
&(Tf)(a, b) \\
&= \int_A \int_B \overline{K(a, b, c, d)} f(c, d) d\mu_B(d) d\mu_A(c) \\
&= \int_{\hat{A}} \int_{\hat{B}} \overline{((\mathcal{F}_A \mathcal{F}_B)_2 \bar{K})(a, b, \alpha, \beta)} (\mathcal{F}_A \mathcal{F}_B f)(\alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \quad (2.3)
\end{aligned}$$

for almost all $(a, b) \in A \times B$, where $(\mathcal{F}_A \mathcal{F}_B)_2 \bar{K}$ is the Fourier transform of $\bar{K}$ with respect to the second ordered pair. Then

$$\begin{aligned}
&(T_\sigma f)(a, b) \\
&= \int_{\hat{A}} \int_{\hat{B}} \left(\int_A \int_B \alpha(c) \beta(d) K(a, b, c, d) d\mu_B(d) d\mu_A(c) \right) (\mathcal{F}_A \mathcal{F}_B f)(\alpha, \beta) \times \\
&\quad \times d\nu_B(\beta) d\nu_A(\alpha) \\
&= \int_{\hat{A}} \int_{\hat{B}} \alpha(a) \beta(b) \left(\int_A \int_B \overline{\alpha(a) \beta(b)} \alpha(c) \beta(d) K(a, b, c, d) d\mu_B(d) d\mu_A(c) \right) \times \\
&\quad \times (\mathcal{F}_A \mathcal{F}_B f)(\alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \quad (2.4)
\end{aligned}$$

for almost all $(a, b) \in A \times B$. Therefore

$$\begin{aligned}
& \sigma(a, b, \alpha, \beta) \\
&= \int_A \int_B \overline{\alpha(a)\beta(b)} \alpha(c)\beta(d) K(a, b, c, d) d\mu_B(d) d\mu_A(c) \\
&= \overline{\alpha(a)\beta(b)} \int_A \int_B \overline{\alpha(c^{-1})\beta(d^{-1})} K(a, b, c, d) d\mu_B(d) d\mu_A(c) \\
&= \overline{\alpha(a)\beta(b)} \int_A \int_B \alpha(c)\beta(d) K(a, b, c, d) d\mu_B(d) d\mu_A(c) \\
&= \overline{\alpha(a)\beta(b)} ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} K)(a, b, \alpha, \beta)
\end{aligned} \tag{2.5}$$

for almost all $(a, b, \alpha, \beta) \in (A \times B) \times (\widehat{A} \times \widehat{B})$. By Plancherel's formula,

$$\begin{aligned}
& \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} |\sigma(a, b, \alpha, \beta)|^2 d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a) \\
&= \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} K)(a, b, \alpha, \beta)|^2 d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a) \\
&= \int_A \int_B \int_A \int_B |K(a, b, c, d)|^2 d\mu_B(d) d\mu_A(c) d\mu(b) d\mu(a).
\end{aligned}$$

So, $\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ and $T = T_\sigma$ with Hilbert-Schmidt norm $\|T_\sigma\|_{S_2(A \times B)}$ given by

$$\begin{aligned}
& \|T_\sigma\|_{S_2(A \times B)} \\
&= \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} |\sigma(a, b, \alpha, \beta)|^2 d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a) \\
&= \|\sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}.
\end{aligned}$$

□

3 Products

Theorem 3.1 *Let σ and τ be functions in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Then $T_\sigma T_\tau = T_\lambda$, where $\lambda \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ is given by*

$$\begin{aligned}
& \lambda(a, b, \alpha, \beta) \\
&= \int_A \int_B \alpha(c)\beta(d) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1}) \tau(c, d, \alpha^{-1}, \beta^{-1}) \times \\
& \quad \times d\mu_B(d) d\mu_A(c)
\end{aligned}$$

for almost all $(a, b, \alpha, \beta) \in (A \times B) \times (\widehat{A} \times \widehat{B})$. Moreover,

$$\|\lambda\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} \leq \|\sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} \|\tau\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}.$$

Proof Let $f \in L^2(A \times B)$. Then by (2.2),

$$(T_\sigma f)(a, b) = \int_A \int_B K_\sigma(a, b, c, d) f(c, d) d\mu_B(d) \mu_A(c)$$

for almost all $(a, b) \in (A \times B)$, where

$$K_\sigma(a, b, c, d) = ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1})$$

for almost all $(a, b), (c, d) \in A \times B$, and

$$(T_\tau f)(a, b) = \int_A \int_B K_\tau(a, b, c, d) f(c, d) d\mu_B(d) d\mu_A(c)$$

for almost all $(a, b) \in A \times B$, where

$$K_\tau(a, b, c, d) = ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \tau)(a, b, a \circ c^{-1}, b \bullet d^{-1})$$

for almost all $(a, b), (c, d) \in A \times B$. Therefore

$$(T_\sigma T_\tau f)(a, b) = \int_A \int_B K(a, b, c, d) f(c, d) d\mu_B(d) d\mu_A(c)$$

for almost all $(a, b) \in A \times B$, where

$$\begin{aligned} K(a, b, c, d) &= \int_A \int_B ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ x^{-1}, b \bullet y^{-1}) \times \\ &\quad \times H(x, y, c, d) d\mu_B(y) d\mu_A(x) \end{aligned}$$

and

$$H(x, y, c, d) = ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \tau)(x, y, x \circ c^{-1}, y \bullet d^{-1})$$

for almost all $(x, y), (c, d) \in (A \times B)$. By (2.5), the symbol λ of $T_\sigma T_\tau$ is given by

$$\begin{aligned} &\lambda(a, b, \alpha, \beta) \\ &= \frac{\overline{\alpha(a)\beta(b)}}{\alpha(a)\beta(b)} \int_A \int_B ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ x^{-1}, b \bullet y^{-1}) \times \\ &\quad \times ((\mathcal{F}_A \mathcal{F}_B)_2 H)(x, y, \alpha, \beta) d\mu_B(y) d\mu_A(x) \end{aligned}$$

for almost all $(a, b, \alpha, \beta) \in (A \times B) \times (\widehat{A} \times \widehat{B})$. But

$$\begin{aligned} & ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} H)(x, y, \alpha, \beta) \\ &= \int_A \int_B \alpha(c) \beta(d) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \tau)(x, y, x \circ c^{-1}, y \bullet d^{-1}) \times \\ & \quad \times d\mu_B(d) d\mu_A(c) \end{aligned}$$

for almost all $(x, y, \alpha, \beta) \in (A \times B) \times (\widehat{A} \times \widehat{B})$. Let $u = x \circ c^{-1}$ and $v = y \bullet d^{-1}$. Then we have

$$\begin{aligned} & ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} H)(x, y, \alpha, \beta) \\ &= \int_A \int_B \alpha(x \circ u^{-1}) \beta(y \bullet v^{-1}) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \tau)(x, y, u, v) \times \\ & \quad \times d\mu_B(v^{-1}) d\mu_A(u^{-1}) \\ &= \alpha(x) \beta(y) \int_A \int_B \alpha(u^{-1}) \beta(v^{-1}) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \tau)(x, y, u, v) \times \\ & \quad \times d\mu_B(v^{-1}) d\mu_A(u^{-1}) \\ &= \alpha(x) \beta(y) \int_A \int_B \alpha(u) \beta(v) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \tau)(x, y, u^{-1}, v^{-1}) \times \\ & \quad \times d\mu_B(v) d\mu_A(u) \\ &= \alpha(x) \beta(y) \tau(x, y, \alpha^{-1}, \beta^{-1}) \end{aligned}$$

for almost all $(x, y), (\alpha, \beta) \in (A \times B) \times (\widehat{A} \times \widehat{B})$. Therefore

$$\begin{aligned} & \lambda(a, b, \alpha, \beta) \\ &= \int_A \int_B \alpha(x) \beta(y) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ u, b \bullet v) \tau(u, v, \alpha^{-1}, \beta^{-1}) \times \\ & \quad \times d\mu_B(v) d\mu_A(u) \end{aligned}$$

for almost all $(a, b, \alpha, \beta) \in (A \times B) \times (\widehat{A} \times \widehat{B})$. Now, by Minkowski's inequality in integral form and Fubini's theorem, we get

$$\begin{aligned} & \|\lambda\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} \\ &= \left(\int_{\widehat{A}} \int_{\widehat{B}} \int_A \int_B |\lambda(a, b, \alpha, \beta)|^2 d\mu_B(b) d\mu_A(a) d\nu_B(\beta) d\nu_A(\alpha) \right)^{1/2} \\ &= \left(\int_{\widehat{A}} \int_{\widehat{B}} \int_A \int_B \left| \int_A \int_B (*, *, *) d\mu_B(v) d\mu_A(u) \right|^2 \times \right. \end{aligned}$$

$$\begin{aligned}
& \times d\mu_B(b) d\mu_A(a) \nu_B(\beta) d\nu_A(\alpha) \Big)^{1/2}, \\
\leq & \int_A \int_B \left(\int_{\hat{A}} \int_{\hat{B}} \int_A \int_B |(*, *, *)|^2 d\mu_B(b) d\mu_A(a) d\nu_B(\beta) d\nu_A(\alpha) \right)^{1/2} \times \\
& \times d\mu_B(v) d\mu_A(u),
\end{aligned}$$

where

$$|(*, *, *)|^2 = |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ u, b \bullet v) \tau(u, v, \alpha^{-1}, \beta^{-1})|^2.$$

So,

$$\begin{aligned}
& \|\lambda\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))} \\
= & \int_A \int_B \left[\int_{\hat{A}} \int_{\hat{B}} |\tau(u, v, \alpha^{-1}, \beta^{-1})| \times \right. \\
& \times \left(\int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ x^{-1}, b \bullet y^{-1})|^2 \times \right. \\
& \times d\mu_B(b) d\mu_A(a) \Big) d\nu_B(\beta) d\nu_A(\alpha) \Big]^{1/2} d\mu_B(y) d\mu_A(x) \\
= & \int_A \int_B \left[\int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ x^{-1}, b \bullet d^{-1})|^2 d\mu_B(b) d\mu_A(a) \right]^{1/2} \times \\
& \times \left[\int_{\hat{A}} \int_{\hat{B}} |\tau(u, v, \alpha^{-1}, \beta^{-1})|^2 d\nu_A(\alpha) d\nu_B(b) \right]^{1/2} d\mu_B(y) d\mu_A(x).
\end{aligned}$$

Using the Schwarz inequality and Fubini's theorem,

$$\begin{aligned}
& \|\lambda\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))} \\
\leq & \left[\int_A \int_B \int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1})|^2 \times \right. \\
& \times d\mu_B(y) d\mu_A(x) d\mu_B(b) d\mu_A(a) \Big]^{1/2} \\
\times & \left[\int_A \int_B \int_{\hat{A}} \int_{\hat{B}} |\tau(u, v, \alpha^{-1} \beta^{-1})|^2 \times \right. \\
& \times d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(v) d\nu_A(u) \Big]^{1/2}.
\end{aligned}$$

Let $u = a \circ c^{-1}$ and $v = b \bullet d^{-1}$. Then

$$\|\lambda\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))}$$

$$\begin{aligned}
&\leq \left[\left(\int_A \int_B \int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma(a, b, u, v)|^2 \times \right. \right. \\
&\quad \left. \left. \times d\mu_B(v) d\mu_A(u) d\mu_B(b) d\mu_A(a) \right)^{1/2} \right] \|\tau\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))} \\
&= \|\sigma\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))} \|\tau\|_{L^2((A \times B) \times (\hat{A} \times \hat{B}))}.
\end{aligned}$$

□

4 Adjoints

The following theorem gives a formula for the symbol of the adjoint of a pseudo-differential operator with symbol in $L^2((A \times B) \times (\hat{A} \times \hat{B}))$.

Theorem 4.1 *Let σ be a symbol in $L^2((A \times B) \times (\hat{A} \times \hat{B}))$. Then the adjoint of the pseudo-differential operator $T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B)$ is the pseudo-differential operator $T_\sigma^* : L^2(A \times B) \rightarrow L^2(A \times B)$ such that $T_\sigma^* = T_\tau$ and τ is the function in $L^2((A \times B) \times (\hat{A} \times \hat{B}))$ given by*

$$\begin{aligned}
&\tau(a, b, \alpha, \beta) \\
&= \overline{\alpha(a)\beta(b)} \int_A \int_B \alpha(c)\beta(d)((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(c, d, a \circ c^{-1}, b \bullet d^{-1}) d\mu_B(d) d\mu_A(c)
\end{aligned}$$

for almost all $(a, b) \in A \times B$ and $(\alpha, \beta) \in \hat{A} \times \hat{B}$.

Proof Let $\sigma \in L^2((A \times B) \times (\hat{A} \times \hat{B}))$. Then by (2.2), $T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B)$ is the Hilbert–Schmidt operator with kernel K given by

$$K(a, b, c, d) = ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1})$$

for almost all (a, b) and (c, d) in $A \times B$. Thus, the kernel H of $T_\sigma^* : L^2(A \times B) \rightarrow L^2(A \times B)$ is the function on $(A \times B) \times (A \times B)$ given by

$$H(a, b, c, d) = \overline{((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(c, d, c \circ a^{-1}, d \bullet b^{-1})}$$

for almost all (a, b) and (c, d) in $A \times B$. By (2.5), the symbol τ of $T_\sigma^* : L^2(A \times B) \rightarrow L^2(A \times B)$ is given by

$$\tau(a, b, \alpha, \beta) = \overline{\alpha(a)\beta(b)}((\mathcal{F}_A \mathcal{F}_B)_2 H)(a, b, \alpha^{-1}, \beta^{-1}) \quad (4.6)$$

for almost all $(a, b) \in A \times B$ and $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. We have

$$\begin{aligned}
& ((\mathcal{F}_A \mathcal{F}_B)_2 H)(a, b, \alpha^{-1}, \beta^{-1}) \\
&= \int_A \int_B \overline{\alpha^{-1}(c) \beta^{-1}(d)} H(a, b, c, d) d\mu_B(d) \mu_A(c) \\
&= \int_A \int_B \alpha(c) \beta(d) \overline{((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(c, d, c \circ a^{-1}, d \bullet b^{-1})} d\mu_B(d) d\mu_A(c)
\end{aligned} \tag{4.7}$$

for almost all $(a, b) \in A \times B$ and $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. and

$$\begin{aligned}
& \overline{((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(c, d, c \circ a^{-1}, d \bullet b^{-1})} \\
&= \int_{\widehat{A}} \int_{\widehat{B}} \overline{\alpha(c \circ a^{-1}) \beta(d \bullet b^{-1})} \overline{\sigma}(c, d, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \\
&= \int_{\widehat{A}} \int_{\widehat{B}} \alpha(a \circ c^{-1}) \beta(b \bullet d^{-1}) \overline{\sigma}(c, d, \alpha, \beta) d\nu_B(d) d\nu_A(\alpha) \\
&= ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \overline{\sigma})(c, d, a \circ c^{-1}, b \bullet d^{-1})
\end{aligned} \tag{4.8}$$

for almost all (a, b) and (c, d) in $A \times B$. By (4.6), (4.7) and (4.8), we get

$$\begin{aligned}
& \tau(a, b, \alpha, \beta) \\
&= \overline{\alpha(a) \beta(b)} \int_A \int_B \alpha(c) \beta(d) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \overline{\sigma})(c, d, a \circ c^{-1}, b \bullet d^{-1}) d\mu_B(d) d\mu_A(c)
\end{aligned}$$

for almost all $(a, b) \in A \times B$ and $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. To prove that $\tau \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$, we change the variable of integration from (c, d) to (c', d') via

$$(c', d') = (a \circ c^{-1}, b \bullet d^{-1})$$

to get

$$\begin{aligned}
& \tau(a, b, \alpha, \beta) \\
&= \int_A \int_B \overline{\alpha(c') \beta(d')} ((\mathcal{F}_A \mathcal{F}_B)^{-1} \overline{\sigma})(a \circ c'^{-1}, b \bullet d'^{-1}, c', d') d\mu_B(d') d\mu_A(c')
\end{aligned} \tag{4.9}$$

By Plancherel's theorem and (4.9),

$$\|\tau\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}^2$$

$$\begin{aligned}
&= \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} |\tau(a, b, \alpha, \beta)|^2 \times \\
&\quad \times d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a) \\
&= \int_A \int_B \int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \bar{\sigma})(a \circ c'^{-1}, b \bullet d'^{-1}, c', d')|^2 \times \\
&\quad \times d\mu_B(d') d\mu_A(c') d\mu_B(b) d\mu_A(a).
\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}
&\|\tau\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}^2 \\
&= \int_A \int_B \int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \bar{\sigma})(a \circ c', b \bullet d'^{-1}, c', d')|^2 \times \\
&\quad \times d\mu_B(b) d\mu_A(a) d\mu_B(d') d\mu_A(c') \\
&= \int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \bar{\sigma})(a, b, c', d')|^2 \times \\
&\quad \times d\mu_B(b) d\mu_A(a) d\mu_B(d') d\mu_A(c') \\
&= \int_A \int_B \int_A \int_B |((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \bar{\sigma})(a, b, c', d')|^2 \times \\
&\quad \times d\mu_B(d') d\mu_A(c') d\mu_B(b) d\mu_A(a).
\end{aligned}$$

By Plancherel's theorem again,

$$\begin{aligned}
&\|\tau\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}^2 \\
&= \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} |\bar{\sigma}(a, b, \alpha, \beta)|^2 \times \\
&\quad \times d\nu_B(\beta) d\nu_A(a) d\mu_B(b) d\mu_A(a) \\
&= \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} |\sigma(a, b, \alpha, \beta)|^2 \times \\
&\quad \times d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a) \\
&= \|\sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}^2
\end{aligned}$$

and the proof is complete. $\square$

5 The Banach $\ast$ -Algebra $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$

For all functions σ and τ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$, we can use Theorem 3.1 to define the product $\sigma \otimes \tau$ of σ and τ to be the function in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$

by

$$\begin{aligned}
& (\sigma \circledast \tau)(a, b, \alpha, \beta) \\
&= \int_A \int_B \alpha(c) \beta(d) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1}) \tau(c, d, \alpha^{-1}, \beta^{-1}) \times \\
& \quad \times d\mu_B(d) d\mu_A(c)
\end{aligned}$$

for almost all $(a, b) \in A \times B$ and $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. We can also use Theorem 4.1 to define the involution σ^* of σ by

$$\begin{aligned}
& \sigma^*(a, b, \alpha, \beta) \\
&= \overline{\alpha(a) \beta(b)} \int_A \int_B \alpha(c) \beta(d) ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \bar{\sigma})(c, d, a \circ c^{-1}, b \bullet d^{-1}) \times \\
& \quad \times d\mu_B(d) d\mu_A(c)
\end{aligned}$$

for almost all $(a, b) \in A \times B$ and $(\alpha, \beta) \in \widehat{A} \times \widehat{B}$. Then $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ becomes a Banach $*$ -algebra with respect to the product $\circledast$ and the involution $*$. The product $\circledast$ in the Banach $*$ -algebra $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ is the main tool to give a characterization of trace class pseudo-differential operators with symbol in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Motivated by [17], let W be the subspace of all functions σ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ such that there exist functions ω and κ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ for which

$$\sigma = \omega \circledast \kappa.$$

Theorem 5.1 *Let $\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Then $T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B)$ is a trace class pseudo-differential operator if and only if $\sigma \in W$.*

Proof Let $\sigma \in W$. Then there exist functions ω and κ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ such that

$$\sigma = \omega \circledast \kappa.$$

By Theorem 3.1,

$$T_\omega T_\kappa = T_{\omega \circledast \kappa} = T_\sigma.$$

By Theorem 2.2, T_ω and T_κ are Hilbert–Schmidt operators on $L^2(A \times B)$. So, T_σ is a trace class operator on $L^2(A \times B)$. Conversely, suppose that T_σ is a trace class operator on $L^2(A \times B)$. Then there exist Hilbert–Schmidt operators S and T on $L^2(A \times B)$ such that

$$T_\sigma = ST.$$

By Theorem 2.2, we can find functions ω and κ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ such that

$$S = T_\omega$$

and

$$T = T_\kappa.$$

Therefore

$$T_\sigma = T_\omega T_\kappa = T_{\omega \otimes \kappa}.$$

So,

$$\sigma = \omega \otimes \kappa,$$

i.e., $\sigma \in W$. □

Remark 5.2 It is an immediate corollary of Theorem 5.1 that W is a subspace of $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. More information about W is given in the following theorem.

Theorem 5.3 *W is a dense subspace of $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$.*

Proof Let $\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Then by Theorem 2.2, the pseudo-differential operator T_σ is a Hilbert–Schmidt operator on $L^2(A \times B)$. So, there exists a sequence $\{F_j\}_{j=1}^\infty$ of finite rank operators on $L^2(A \times B)$ such that

$$F_j \rightarrow T_\sigma$$

in the space $S_2(A \times B)$ of all Hilbert–Schmidt operators on $L^2(A \times B)$ as $j \rightarrow \infty$. Since F_j is a trace class operator, and hence a Hilbert–Schmidt operator, on $L^2(A \times B)$ for all $j = 1, 2, \dots$, it follows from Theorem 2.2 again that there exists a function $\sigma_j \in W$ such that

$$F_j = T_{\sigma_j}, \quad j = 1, 2, \dots$$

By Theorem 2.2 again,

$$\|\sigma_j - \sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} = \|F_j - T_\sigma\|_{S_2(A \times B)} \rightarrow 0$$

as $j \rightarrow \infty$. Therefore W is dense in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. □

Remark 5.4 The dense subspace W of $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ is best dubbed the trace class subspace of $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ because W consists of all symbols σ defined on $(A \times B) \times (\widehat{A} \times \widehat{B})$ for which T_σ is a trace class operator on $L^2(A \times B)$.

The following theorem gives a formula for the trace of a trace class pseudo-differential operator with symbol in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. More succinctly, it gives the trace of every pseudo-differential operator T_σ on $L^2(A \times B)$ with symbol σ in W .

Theorem 5.5 *Let $\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ be such that $T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B)$ is a trace class pseudo-differential operator. Then the trace $\text{tr}(T_\sigma)$ of T_σ is given by*

$$\text{tr}(T_\sigma) = \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} \sigma(a, b, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a).$$

Proof By (2.2), the kernel of $T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B)$ is the function K in $L^2((A \times B) \times (A \times B))$ given by

$$K(a, b, c, d) = ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ c^{-1}, b \bullet d^{-1})$$

for almost all (a, b) and (c, d) in $A \times B$. Therefore

$$\begin{aligned} \text{tr}(T_\sigma) &= \int_A \int_B K(a, b, a, b) d\mu_B(b) d\mu_A(a) \\ &= \int_A \int_B ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, a \circ a^{-1}, b \bullet b^{-1}) d\mu_B(b) d\mu_A(a) \\ &= \int_A \int_B ((\mathcal{F}_A \mathcal{F}_B)_2^{-1} \sigma)(a, b, e_A, e_B) d\mu_B(b) d\mu_A(a), \end{aligned}$$

where e_A is the identity element in A and e_B is the identity element in B . So,

$$\text{tr}(T_\sigma) = \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} \alpha(e_A) \beta(e_B) \sigma(a, b, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a).$$

Since $\alpha(e_A) = \beta(e_B) = 1$, it follows that

$$\text{tr}(T_\sigma) = \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} \sigma(a, b, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) d\mu_B(b) d\mu_A(a).$$

□

6 H^* -Algebras

A H^* -algebra $\mathcal{A}$ is a Banach $*$ -algebra with involution $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ and norm $\| \cdot \|$ that comes from an inner product $(\cdot, \cdot)$ such that

$$(ab, c) = (b, a^*c), \quad a, b, c \in \mathcal{A}.$$

References for H^* -algebras are [1, 2, 6, 7, 11, 14, 16], among others.

Let $S_2(A \times B)$ be the set of all pseudo-differential operators on $L^2(A \times B)$ given by

$$S_2(A \times B) = \{T_\sigma : L^2(A \times B) \rightarrow L^2(A \times B) : \sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))\}.$$

Then we have the following theorem.

Theorem 6.1 $S_2(A \times B)$ is a H^* -algebra with respect to the product

$$S_2(A \times B) \times S_2(A \times B) \ni (T_\sigma, T_\tau) \mapsto T_\sigma T_\tau \in S_2(A \times B),$$

the involution

$$S_2(A \times B) \ni T_\sigma \mapsto T_\sigma^* \in S_2(A \times B)$$

and the inner product $(\cdot, \cdot)_{S_2(A \times B)}$ given by

$$(T_\sigma, T_\tau)_{S_2(A \times B)} = \text{tr}(T_\sigma T_\tau^*)$$

for all σ and τ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$.

Proof To prove that $S_2(A \times B)$ is a Banach $*$ -algebra with involution, we need only prove that $S_2(A \times B)$ is complete with respect to the norm $\| \cdot \|_{S_2(A \times B)}$. Let $\{T_{\sigma_k}\}_{k=1}^\infty$ be a Cauchy sequence in $S_2(A \times B)$. Then by Theorem 2.2, we get

$$\|\sigma_j - \sigma_k\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} = \|T_{\sigma_j} - T_{\sigma_k}\|_{S_2(A \times B)} \rightarrow 0$$

as $j, k \rightarrow \infty$. Therefore $\{\sigma_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Since $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ is complete, we can find a function

$$\sigma \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$$

such that

$$\sigma_k \rightarrow \sigma$$

in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ as $k \rightarrow \infty$. By Theorem 2.2 again,

$$\|T_{\sigma_k} - T_\sigma\|_{S_2(A \times B)} = \|\sigma_k - \sigma\|_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} \rightarrow 0$$

as $k \rightarrow \infty$. So,

$$T_{\sigma_k} \rightarrow T_\sigma$$

in $S_2(A \times B)$ as $k \rightarrow \infty$. Therefore $S_2(A \times B)$ is complete with respect to the norm $\|\cdot\|_{S_2(A \times B)}$. Now, let T_σ , T_τ and T_ω be elements in $S_2(A \times B)$. Then

$$\begin{aligned} (T_\sigma T_\tau, T_\omega)_{S_2(A \times B)} &= \text{tr}((T_\sigma T_\tau) T_\omega^*) \\ &= \text{tr}(T_\sigma (T_\tau T_\omega^*)) \\ &= \text{tr}((T_\tau T_\omega^*) T_\sigma) \\ &= \text{tr}(T_\tau (T_\omega^* T_\sigma)) \\ &= \text{tr}(T_\tau (T_\omega^* T_\sigma^{**})) \\ &= \text{tr}(T_\tau (T_\sigma^* T_\omega)^*) \\ &= (T_\tau, T_\sigma^* T_\omega)_{S_2(A \times B)} \end{aligned}$$

and this completes the proof of the theorem. $\square$

At the level of functions, we equip $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ with the product

$$L^2((A \times B) \times (\widehat{A} \times \widehat{B})) \ni (\sigma, \tau) \mapsto \sigma \otimes \tau \in L^2((A \times B) \times (\widehat{A} \times \widehat{B})),$$

the involution

$$L^2((A \times B) \times (\widehat{A} \times \widehat{B})) \ni \sigma \mapsto \sigma^* \in L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$$

and the inner product $(\cdot, \cdot)_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))}$ given by

$$\begin{aligned} &(\sigma, \tau)_{L^2((A \times B) \times (\widehat{A} \times \widehat{B}))} \\ &= \int_A \int_B \int_{\widehat{A}} \int_{\widehat{B}} (\sigma \otimes \tau^*)(a, b, \alpha, \beta) d\nu_B(\beta) d\nu_A(\alpha) \times \\ &\quad \times d\mu_B(b) d\mu_A(a) \end{aligned}$$

for all σ and τ in $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$. Then $L^2((A \times B) \times (\widehat{A} \times \widehat{B}))$ is a H^* -algebra.

7 The Open Upper Half Space

The open upper half space $\mathbb{H}_+^{n+1}$ is given by

$$\mathbb{H}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+,$$

where

$$\mathbb{R}_+ = (0, \infty).$$

It is the direct product of the additive group $\mathbb{R}^n$ and the multiplicative group $\mathbb{R}_+$. In $\mathbb{R}^n$, the binary operation is given by

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto x + y \in \mathbb{R}^n,$$

and in $\mathbb{R}_+$, the binary operation is

$$\mathbb{R}_+ \times \mathbb{R}_+ \ni (t, s) \mapsto ts \in \mathbb{R}_+.$$

Therefore the binary operation $\cdot$ in the open upper half space $\mathbb{H}_+^{n+1}$ is given by

$$(x, t) \cdot (y, s) = (x + y, ts)$$

for all $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}_+$. The Haar measure on $\mathbb{R}^n$ is the usual Lebesgue measure dx and the Haar measure μ on $\mathbb{R}_+$ is taken to be the one in [14], denoted by μ , where

$$d\mu(t) = \frac{dt}{t}, \quad t \in \mathbb{R}_+.$$

The Haar measure $\mu_{\mathbb{H}_+^{n+1}}$ on $\mathbb{H}_+^{n+1}$ is then given by

$$d\mu_{\mathbb{H}_+^{n+1}}(x, t) = dx \frac{dt}{t}, \quad x \in \mathbb{R}^n, t \in \mathbb{R}_+.$$

The dual group of $\mathbb{R}^n$ is again $\mathbb{R}^n$ and the dual group of $\mathbb{R}_+$ is shown in [14] to be $\mathbb{R}$. So, the Haar measure on the dual group $\widehat{\mathbb{H}_+^{n+1}}$ is the Lebesgue measure $d\xi d\nu$ on

$$\mathbb{R}^{n+1} = \{(\xi, \nu) : \xi \in \mathbb{R}^n, \nu \in \mathbb{R}\}.$$

The Fourier transform on $\mathbb{R}^n$ is the usual Euclidean Fourier transform $\mathcal{F}$ on $\mathbb{R}^n$ given by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n,$$

for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and the Fourier transform on $\mathbb{R}_+$ is the Mellin transform M on $\mathbb{R}_+$ given by

$$(Mf)(\nu) = (2\pi)^{-1/2} \int_0^\infty t^{-i\nu} f(t) \frac{dt}{t}, \quad \nu \in \mathbb{R},$$

for all $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. It should be noted that the measure μ is understood and omitted. As shown in the paper [?], the Fourier transform $\mathcal{F}_{\mathbb{H}_+^{n+1}}$ and the inverse Fourier transform $\mathcal{F}_{\mathbb{H}_+^{n+1}}^{-1}$ on $\mathbb{H}_+^{n+1}$ are given, respectively, by

$$\mathcal{F}_{\mathbb{H}_+^{n+1}} = \mathcal{F}M = M\mathcal{F}$$

and

$$\mathcal{F}_{\mathbb{H}_+^{n+1}}^{-1} = M^{-1}\mathcal{F}^{-1} = \mathcal{F}^{-1}M^{-1}.$$

8 Pseudo-Differential Operators on $L^2(\mathbb{H}_+^{n+1})$ with Symbol in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$

Let σ be a function in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$. Then we define the pseudo-differential operator $T_\sigma : L^2(\mathbb{H}_+^{n+1}) \rightarrow L^2(\mathbb{H}_+^{n+1})$ with symbol σ by

$$(T_\sigma f)(x, t) = (2\pi)^{-(n+1)/2} \int_{-\infty}^\infty \int_{\mathbb{R}^n} e^{ix \cdot \xi} t^{i\nu} \sigma(x, t, \xi, \nu) (\mathcal{F}Mf)(\xi, \nu) d\xi d\nu$$

for all $(x, t) \in \mathbb{H}_+^{n+1}$. By Theorem 1.2 in [?], we have the following theorem.

Theorem 8.1 *A bounded linear operator $A : L^2(\mathbb{H}_+^{n+1}) \rightarrow L^2(\mathbb{H}_+^{n+1})$ is a Hilbert-Schmidt operator if and only if $A = T_\sigma$ with*

$$\sigma \in L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}}).$$

Moreover,

$$\|T_\sigma\|_{S_2(\mathbb{H}_+^{n+1})} = \|\sigma\|_{L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})}.$$

9 Products and Adjoints

We have the following theorem on the product of two pseudo-differential operators on $L^2(\mathbb{H}_+^{n+1})$ with symbol in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$.

Theorem 9.1 *Let σ and τ be functions in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$. Then $T_\sigma T_\tau$ is the pseudo-differential operator on $L^2(\mathbb{H}_+^{n+1})$ with symbol $\sigma \circledast \tau$ given by*

$$\begin{aligned} & (\sigma \circledast \tau)(x, t, \xi, \nu) \\ = & (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^n} \int_0^\infty e^{ix \cdot \xi} t^{i\nu} ((\mathcal{F}M)_2^{-1} \sigma) \left(x, t, x - y, \frac{t}{s} \right) \tau(y, s, -\xi, -\nu) \times \\ & \times \frac{ds}{s} dy \end{aligned} \quad (9.10)$$

for almost all (x, t, ξ, ν) in $\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}}$, where $(\mathcal{F}M)_2^{-1} \sigma$ denotes the inverse Fourier transform of σ with respect to the second pair of coordinates.

As for adjoints, we have the following theorem.

Theorem 9.2 *Let σ be a function in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$. Then the adjoint T_σ^* of T_σ on $L^2(\mathbb{H}_+^{n+1})$ is the pseudo-differential operator on $L^2(\mathbb{H}_+^{n+1})$ with symbol σ^* in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$ given by*

$$\begin{aligned} & \sigma^*(x, t, \xi, \nu) \\ = & (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^n} \int_0^\infty e^{-i(x-y) \cdot \xi} \left(\frac{s}{t} \right)^{i\nu} ((\mathcal{F}M)_2^{-1} \bar{\sigma}) \left(y, s, x - y, \frac{t}{s} \right) \frac{ds}{s} dy \end{aligned} \quad (9.11)$$

for almost all (x, t, ξ, ν) in $\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}}$.

10 The Banach $*$ -Algebra $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$ and Trace Class Operators on $L^2(\mathbb{H}_+^{n+1})$

Let σ and τ be functions in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$. If we define the product $\sigma \circledast \tau$ of σ and τ and the adjoint σ^* of σ by, respectively, (9.10) and (9.11), then

$L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$ is a Banach $*$ -algebra with respect to the product $\otimes$ and the involution $*$.

We can now look at the subspace W of $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$ consisting of all functions σ in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$ such that σ has a factorization

$$\sigma = \omega \otimes \kappa$$

with factors ω and κ in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$. Then we have the following characterization of trace class pseudo-differential operators on $L^2(\mathbb{H}_+^{n+1})$.

Theorem 10.1 *Let $\sigma \in L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$. Then $T_\sigma : L^2(\mathbb{H}_+^{n+1}) \rightarrow L^2(\mathbb{H}_+^{n+1})$ is a trace class operator if and only if $\sigma \in W$.*

Another theorem in connection with the space W is given by

Theorem 10.2 *For every pseudo-differential operator T_σ on $L^2(\mathbb{H}_+^{n+1})$ with symbol σ in W , we have*

$$\text{tr}(T_\sigma) = \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \int_{-\infty}^\infty \sigma(x, t, \xi, \nu) d\nu d\xi \frac{dt}{t} dx.$$

11 A H^* -Algebra of Bounded Linear Operators on $L^2(\mathbb{H}_+^{n+1})$ and the H^* -algebra $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$

Let $S_2(\mathbb{H}_+^{n+1})$ be the set of all pseudo-differential operators on $L^2(\mathbb{H}_+^{n+1})$ given by

$$S_2(\mathbb{H}_+^{n+1}) = \{T_\sigma : \sigma \in L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})\}.$$

Then $S_2(\mathbb{H}_+^{n+1})$ is a H^* -algebra with respect to the product

$$S_2(\mathbb{H}_+^{n+1}) \times S_2(\mathbb{H}_+^{n+1}) \ni (T_\sigma, T_\tau) \mapsto T_\sigma T_\tau \in S_2(\mathbb{H}_+^{n+1}),$$

the involution

$$S_2(\mathbb{H}_+^{n+1}) \ni T_\sigma \mapsto T_\sigma^* \in S_2(\mathbb{H}_+^{n+1})$$

and the inner product $(\cdot, \cdot)_{S_2(\mathbb{H}_+^{n+1})}$ given by

$$(T_\sigma, T_\tau)_{S_2(\mathbb{H}_+^{n+1})} = \text{tr}(T_\sigma T_\tau^*)$$

for all σ and τ in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$.

We can now give the H^* -algebra $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$.

Theorem 11.1 $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$ is a H^* -algebra with respect to the product $\sigma \otimes \tau$ of σ and τ , the involution σ^* of σ given, respectively, by (9.10) and (9.11), and the inner product given by

$$\begin{aligned} & (\sigma, \tau)_{L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})} \\ &= \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \int_{-\infty}^\infty (\sigma \otimes \tau^*)(x, t, \xi, \nu) d\nu d\xi \frac{dt}{t} dx \end{aligned}$$

for almost all σ and τ in $L^2(\mathbb{H}_+^{n+1} \times \widehat{\mathbb{H}_+^{n+1}})$.

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