

On Evolution of the Classical Heat and the Quantum Wave Generated by the Laplacian on the Heisenberg Group

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Abstract: The Laplacian on the Heisenberg group as a sum of squares of the standard vector fields on the Heisenberg group is first made explicit to be a Laplace-Beltrami operator on the Heisenberg group, which is well known to be a Riemannian manifold. The dissipation of the heat generated by the Laplacian on the Heisenberg group is then given in terms of estimates reminiscent of the Hausdorff–Young inequalities for the Euclidean Fourier transform. The evolution of the quantum wave driven by the Laplacian on the Heisenberg group is given. Weyl transforms, which are pseudo-differential operators in disguise, are instrumental in deriving the explicit formula for the heat kernel of the Laplacian on the Heisenberg group.

2020 Mathematics Subject Classification: Primary: 35R03

Key Words: Heisenberg group, vector field, Laplacian, ellipticity, Laplace–Beltrami operator, Riemannian manifold, Riemannian metric, scalar curvature, heat, dissipation, Riesz–Thorin theorem, Schrödinger equation, quantum wave, Stone’s theorem, pseudo-differential operator, Weyl transform

1 The Laplacian on the Heisenberg Group

For the sake of simplicity and transparency in notation, we look at the one-dimensional Heisenberg group only. Generalizations to higher-dimensional Heisenberg groups are routine and hence omitted.

We identify points in $\mathbb{R}^2$ with points in $\mathbb{C}$ via the identification

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C}.$$

Let $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$. Then for all points (z, t) and (w, s) in $\mathbb{H}^1$, we define $(z, t) \cdot (w, s)$ by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{4}[z, w] \right),$$

where $[z, w]$ is the symplectic form of z and w given by

$$[z, w] = 2 \operatorname{Im}(z\bar{w}).$$

In fact, $\mathbb{H}^1$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz dt$.

Let X , Y , and T be vector fields on $\mathbb{H}^1$ given by

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x \frac{\partial}{\partial t},$$

and

$$T = \frac{\partial}{\partial t}.$$

They are known as the standard vector fields on the Heisenberg group. Then the Laplacian $\Delta_{\mathbb{H}^1}$ on $\mathbb{H}^1$ is defined by

$$\Delta_{\mathbb{H}^1} = -(X^2 + Y^2 + T^2). \quad (1.1)$$

To see the Laplacian $\Delta_{\mathbb{H}^1}$ more explicitly, we note that

$$\Delta_{\mathbb{H}^1} = -\Delta - \frac{1}{4}(x^2 + y^2) \frac{\partial^2}{\partial t^2} + \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The ellipticity of $\Delta_{\mathbb{H}^1}$ on $\mathbb{R}^3$ can best be seen by expressing the symbol $\sigma(\Delta_{\mathbb{H}^1})$ of $\Delta_{\mathbb{H}^1}$ as

$$\sigma(\Delta_{\mathbb{H}^1})(x, y, t, \xi, \eta, \tau) = \left(\xi + \frac{1}{2}y\tau \right)^2 + \left(\eta - \frac{1}{2}x\tau \right)^2 + \tau^2$$

for all (x, y, t) and (ξ, η, τ) in $\mathbb{R}^3$. Therefore for a fixed but arbitrary point (x, y, t) in $\mathbb{R}^3$,

$$\sigma(\Delta_{\mathbb{H}^1})(x, y, t, \xi, \eta, \tau) = 0$$

implies that

$$\begin{aligned} \left(\xi + \frac{1}{2}y\tau \right)^2 &= 0, \\ \left(\eta - \frac{1}{2}x\tau \right)^2 &= 0, \end{aligned}$$

and

$$\tau = 0.$$

Therefore $(\xi, \eta, \tau) = (0, 0, 0)$ and $\Delta_{\mathbb{H}^1}$ is elliptic everywhere on $\mathbb{R}^3$. It is worth mentioning that $\Delta_{\mathbb{H}^1}$ is not even a pseudo-differential operator with symbol in the simplest Hörmander class detailed in [12]. To see this, suppose that it is. Since the order of $\Delta_{\mathbb{H}^1}$ is 2, it follows that there exists a positive constant C such that

$$|\sigma(\Delta_{\mathbb{H}^1})(x, y, t, \xi, \eta, \tau)| \leq C(1 + \xi^2 + \eta^2 + \tau^2), \quad (\xi, \eta, \tau) \in \mathbb{R}^3.$$

Therefore for a fixed but arbitrary point (ξ, η, τ) in $\mathbb{R}^3$,

$$\left(\xi + \frac{1}{2}y\tau \right)^2 + \left(\eta - \frac{1}{2}x\tau \right)^2 + \tau^2 \leq (1 + \xi^2 + \eta^2 + \tau^2)$$

and this is a contradiction if we let $x \rightarrow \infty$ or $y \rightarrow \infty$.

In fact, $\Delta_{\mathbb{H}^1}$ is the Laplace–Beltrami operator corresponding to the Riemannian metric g on $\mathbb{R}^3$, where

$$g(x, y, t) = \begin{bmatrix} 1 + (y^2/4) & -xy/4 & -y/2 \\ -xy/4 & 1 + (x^2/4) & x/2 \\ -y/2 & x/2 & 1 \end{bmatrix}, \quad (x, y, t) \in \mathbb{R}^3.$$

Equivalently, $\mathbb{H}^1$ is the Riemannian manifold $\mathbb{R}^3$ equipped with the metric g . This fact can be expressed by the formula

$$\Delta_{\mathbb{H}^1} f = -\frac{1}{\sqrt{\det g}} \sum_{j,k=1}^3 \partial_k (g^{jk} \sqrt{\det g} \partial_j f)$$

for all f in $C^2(\mathbb{H}^1)$, where g^{jk} is the entry in the j^{th} row and k^{th} column of the inverse g^{-1} of the matrix g ,

$$\partial_1 = \frac{\partial}{\partial x},$$

$$\partial_2 = \frac{\partial}{\partial y},$$

and

$$\partial_3 = \frac{\partial}{\partial t}.$$

Despite its simplicity, a proof that the Laplacian $\Delta_{\mathbb{H}^1}$ is indeed the Laplace–Beltrami operator on $\mathbb{R}^3$ as a Riemannian manifold is proved in Section 2. Thus, the role of the Laplacian $\Delta_{\mathbb{H}^1}$ on the Heisenberg group $\mathbb{H}^1$ is analogous to, albeit a special case of, that of the Laplace–Beltrami operator on a Riemannian manifold. See, for instance, [8] for more in-depth information about the Laplace–Beltrami operator on a Riemannian manifold. That the Heisenberg group is an important Riemannian manifold in its own right can be seen from [2]. It is a Riemannian manifold with constant scalar curvature $\frac{1}{2}$.

We are interested in finding a solution u of the initial value problem for the heat equation given by

$$\begin{cases} \frac{\partial u}{\partial \rho}(z, t, \rho) = -(\Delta_{\mathbb{H}^1} u)(z, t, \rho), & (z, t) \in \mathbb{R}^3, \rho > 0, \\ u(z, t, 0) = f(z, t), & (z, t) \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

where f is a given suitable function on $\mathbb{R}^3$. In particular, we seek a solution u of the form

$$u(z, t, \rho) = (f *_{\mathbb{H}^1} K_\rho)(z, t) = \int_{-\infty}^{\infty} \int_{\mathbb{C}} f((z, t) \cdot (w, s)^{-1}) K_\rho(w, s) dw ds,$$

where K_ρ is known as the heat kernel of the Laplacian $\Delta_{\mathbb{H}^1}$ on the Heisenberg group $\mathbb{H}^1$. We denote this solution u by $e^{-\rho \Delta_{\mathbb{H}^1}} f$ and we call $\{e^{-\rho \Delta_{\mathbb{H}^1}} : \rho > 0\}$

the *classical heat* generated by the Laplacian on the Heisenberg group. Due to the lack of the abovementioned type of ellipticity of $\Delta_{\mathbb{H}^1}$, the techniques in Chapter 19 of the book [12] cannot even be applied to show that a solution exists.

Using the essential self-adjointness of $\Delta_{\mathbb{H}^1}$, to be proved in Section 6, we can look at the Schrödinger equation governed by the Laplacian $\Delta_{\mathbb{H}^1}$ on $\mathbb{R}^3$ given by

$$\begin{cases} \frac{\partial u}{\partial \rho}(z, t, \rho) = -i(\Delta_{\mathbb{H}^1} u)(z, t, \rho), & (z, t) \in \mathbb{R}^3, \rho \in (-\infty, \infty), \\ u(z, t, 0) = f(z, t), & (z, t) \in \mathbb{R}^3, \end{cases} \quad (1.3)$$

where $f \in L^2(\mathbb{R}^3)$. It is natural to denote this solution u by $e^{-i\rho\Delta_{\mathbb{H}^1}} f$ and we call $\{e^{-i\rho\Delta_{\mathbb{H}^1}} : -\infty < \rho < \infty\}$ the *quantum wave* driven by the Laplacian on the Heisenberg group.

Using the same techniques in Chapters 17–23 of [13], we first give a recall of the inverse Fourier transform of the Laplacian $\Delta_{\mathbb{H}^1}$ with respect to the center of the Heisenberg group $\mathbb{H}^1$. More precisely, for all functions f in $L^2(\mathbb{H}^1)$ and all positive numbers ρ ,

$$(e^{-\rho\Delta_{\mathbb{H}^1}} f)^\tau = e^{-\rho\tilde{L}_\tau} f^\tau,$$

where

$$f^\tau(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-it\tau} f(z, t) dt, \quad z \in \mathbb{C},$$

$$\tilde{L}_\tau = L_\tau + \tau^2,$$

and

$$L_\tau = -\Delta + \frac{1}{4}(x^2 + y^2)\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau$$

for all $\tau \in \mathbb{R} \setminus \{0\}$. By Theorem 22.4 in [13],

$$K_\rho^\tau(z) = (2\pi)^{-1/2} \frac{1}{4\pi \sinh(\tau\rho)} \frac{e^{-\rho\tau^2}\tau}{e^{-\frac{1}{4}\tau|z|^2 \coth(\tau\rho)}}$$

for all z in $\mathbb{C}$, all positive numbers ρ , and all numbers τ in $\mathbb{R} \setminus \{0\}$. So,

$$K_\rho(z, t) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{-it\tau} \frac{\tau}{\sinh(\tau\rho)} e^{-|\tau|^2\rho} e^{-\frac{1}{4}\tau|z|^2 \coth(\tau\rho)} d\tau \quad (1.4)$$

for all points (z, t) in $\mathbb{H}^1$. It should be emphasized that the calculus of pseudo-differential operators, dressed up as Weyl transforms in [11], is instrumental in the derivation of the formula for the heat kernel given in (1.4). All details are given in [13].

The aim of this paper is to obtain a mathematical formulation of the dissipation of the heat driven by the Laplacian on the Heisenberg group in terms of L^p - $L^{p'}$ estimates with $1 \leq p \leq 2$. These estimates can be considered to be the analogs of the Hausdorff–Young inequalities for the Fourier transform on $\mathbb{R}^n$. See, for example, [9] in this connection. For the closely related twisted Laplacians, much more general L^p - L^q estimates have been obtained even for Heisenberg groups with much higher complexities in [14, 15]. In addition, the evolution of the quantum wave driven by $\Delta_{\mathbb{H}^1}$ is given in Section 6.

2 A Laplace–Beltrami Operator on $\mathbb{R}^3$

We prove in this section that $\Delta_{\mathbb{H}^1}$ is a Laplace–Beltrami operator on $\mathbb{R}^3$.

Theorem 2.1 *Let g be the Riemannian metric on $\mathbb{R}^3$ given by*

$$g(x, y, t) = \begin{bmatrix} 1 + (y^2/4) & -xy/4 & -y/2 \\ -xy/4 & 1 + (x^2/4) & x/2 \\ -y/2 & x/2 & 1 \end{bmatrix}, \quad (x, y, t) \in \mathbb{R}^3.$$

Then

$$\Delta_{\mathbb{H}^1} = -\frac{1}{\sqrt{\det g}} \sum_{j,k=1}^3 \partial_k (g^{jk} \sqrt{\det g} \partial_j),$$

where g^{jk} is the entry in the j^{th} row and k^{th} column of the inverse g^{-1} of the matrix g .

Proof By standard computations,

$$\det g(x, y, t) = 1, \quad (x, y, t) \in \mathbb{R}^3,$$

and

$$g^{-1}(x, y, t) = \begin{bmatrix} 1 & 0 & y/2 \\ 0 & 1 & -x/2 \\ y/2 & -x/2 & 1 + ((x^2 + y^2)/4) \end{bmatrix}, \quad (x, y, t) \in \mathbb{R}^3.$$

Therefore

$$\begin{aligned} \partial_1(g^{11}\partial_1) &= \partial_1^2, \\ \partial_1(g^{21}\partial_2) &= 0, \\ \partial_1(g^{31}\partial_3) &= -\frac{y}{2}\partial_1\partial_3, \\ \partial_2(g^{12}\partial_1) &= 0, \\ \partial_2(g^{22}\partial_2) &= \partial_2^2, \\ \partial_2(g^{32}\partial_3) &= \frac{x}{2}\partial_2\partial_3, \\ \partial_3(g^{31}\partial_1) &= -\frac{y}{2}\partial_1\partial_3, \\ \partial_3(g^{32}\partial_2) &= \frac{x}{2}\partial_2\partial_3, \end{aligned}$$

and

$$\partial_3(g^{33}\partial_3) = \left(1 + \frac{x^2 + y^2}{4}\right) \partial_3.$$

Therefore

$$\begin{aligned} & -\frac{1}{\sqrt{\det g}} \sum_{j,k=1}^3 \partial_k(g^{jk}\sqrt{\det g}\partial_j) \\ &= -(\partial_1^2 + \partial_2^2) - \frac{1}{4}(x^2 + y^2)\partial_3^2 + (x\partial_2 - y\partial_1)\partial_3 - \partial_3^2 \\ &= -\Delta - \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2} \\ &= \Delta_{\mathbb{H}^1}. \end{aligned}$$

□

3 L^1 - L^∞ Dissipation

We begin with a lemma.

Lemma 3.1 *For every positive number ρ , $K_\rho \in L^\infty(\mathbb{H}^1)$. In fact,*

$$|K_\rho(z, t)| \leq \frac{1}{2^{5/3}\pi^{3/2}\rho^{3/2}}, \quad (z, t) \in \mathbb{H}^1, \rho > 0.$$

Proof Let ρ be a positive number. Then $\tau \coth(\tau\rho)$ is an even and positive function of τ on $(-\infty, \infty)$. Therefore

$$e^{-\frac{1}{4}\tau|z|^2 \coth(\tau\rho)} \leq 1, \quad (z, t) \in \mathbb{H}^1.$$

Since

$$\sup_{x \in \mathbb{R}} \left| \frac{x}{\sinh x} \right| = 1,$$

it follows that for every positive number ρ ,

$$|K_\rho(z, t)| \leq \frac{1}{8\pi^2\rho} \int_{-\infty}^{\infty} \left| \frac{\rho\tau}{\sinh(\rho\tau)} \right| e^{-\rho\tau^2} d\tau \leq \frac{1}{8\pi^2\rho} \int_{-\infty}^{\infty} e^{-\rho\tau^2} d\tau, \quad (z, t) \in \mathbb{H}^1.$$

Since

$$\int_{-\infty}^{\infty} e^{-\rho\tau^2} d\tau = \frac{1}{\rho^{1/2}} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \left(\frac{2\pi}{\rho} \right)^{1/2},$$

we obtain

$$|K_\rho(z, t)| \leq \frac{1}{2^{5/2}\pi^{3/2}\rho^{3/2}}, \quad (z, t), \rho > 0.$$

□

We can now give the L^1 - L^∞ estimate for the heat generated by the Laplacian $\Delta_{\mathbb{H}^1}$ on the Heisenberg group $\mathbb{H}^1$.

Theorem 3.2 *For every positive number ρ and for every f in $L^1(\mathbb{H}^1)$,*

$$\|e^{-\rho\Delta_{\mathbb{H}^1}} f\|_{L^\infty(\mathbb{H}^1)} \leq \frac{1}{2^{5/2}\pi^{3/2}\rho^{3/2}} \|f\|_{L^1(\mathbb{H}^1)}.$$

Proof Let $f \in L^1(\mathbb{H}^1)$. Then

$$(e^{-\rho\Delta_{\mathbb{H}^1}} f)(z, t) = \int_{-\infty}^{\infty} \int_{\mathbb{C}} f((z, t) \cdot (w, s)^{-1}) K_\rho(w, s) dw ds$$

for all positive numbers ρ and all points (z, t) in $\mathbb{H}^1$. Using Lemma 3.1 and the right invariance of the Haar measure on $\mathbb{H}^1$, we get

$$\begin{aligned} \|e^{-\rho\Delta_{\mathbb{H}^1}}\|_{L^\infty(\mathbb{H}^1)} &\leq \frac{1}{(2^{5/3}\pi\rho)^{3/2}} \int_{-\infty}^{\infty} \int_{\mathbb{C}} |f((z, t) \cdot (w, s)^{-1})| dw ds \\ &= \frac{1}{(2^{5/3}\pi\rho)^{3/2}} \int_{-\infty}^{\infty} \int_{\mathbb{C}} |f(w, s)| dw ds \\ &= \frac{1}{(2^{5/3}\pi\rho)^{3/2}} \|f\|_{L^1(\mathbb{H}^1)}. \end{aligned}$$

□

4 L^2 - L^2 Dissipation

The L^2 - L^2 dissipation of the heat $\{e^{-\rho\Delta_{\mathbb{H}^1}} : \rho > 0\}$ is given in the following theorem.

Theorem 4.1 *For all positive numbers ρ and all functions f in $L^2(\mathbb{H}^1)$,*

$$\|e^{-\rho\Delta_{\mathbb{H}^1}} f\|_{L^2(\mathbb{H}^1)} \leq \frac{1}{2^{5/2}\pi^{3/2}\rho} \|f\|_{L^2(\mathbb{H}^1)}.$$

Proof Let ρ be a positive number and let f be a function in $L^2(\mathbb{H}^1)$. Then by Fubini's theorem and the Plancherel theorem for the Fourier transform, we get

$$\begin{aligned} \|e^{-\rho\Delta_{\mathbb{H}^1}} f\|_{L^2(\mathbb{H}^1)}^2 &= \int_{-\infty}^{\infty} \int_{\mathbb{C}} |(e^{-\rho\Delta_{\mathbb{H}^1}} f)^\tau(z)|^2 dz d\tau \\ &= \int_{-\infty}^{\infty} \left(\int_{\mathbb{C}} |(e^{-\rho\tilde{L}_\tau} f^\tau)(z)|^2 dz \right) d\tau \\ &= \int_{-\infty}^{\infty} \|e^{-\rho\tilde{L}_\tau} f^\tau\|_{L^2(\mathbb{C})}^2 d\tau \\ &\leq \frac{1}{32\pi^3} \int_{-\infty}^{\infty} \frac{e^{-2\tau^2\rho\tau^2}}{\sinh^2(|\tau|\rho)} \left(\int_{\mathbb{C}} |f^\tau(z)|^2 dz \right) d\tau \\ &= \frac{1}{32\pi^3} \int_{-\infty}^{\infty} \frac{e^{-2\tau^2\rho\tau^2}}{\sinh^2(|\tau|\rho)} \left(\int_{\mathbb{C}} |\check{f}(z, \tau)|^2 dz \right) d\tau. \end{aligned}$$

Since

$$\sup_{x \in \mathbb{R}} \left| \frac{x}{\sinh x} \right| = 1,$$

it follows that

$$\|e^{-\rho \Delta_{\mathbb{H}^1}} f\|_{L^2(\mathbb{H}^1)}^2 \leq \frac{1}{32\pi^3 \rho^2} \int_{\mathbb{C}} \int_{-\infty}^{\infty} |f(z, t)|^2 dz dt = \frac{1}{32\pi^3 \rho^2} \|f\|_{L^2(\mathbb{H}^1)}^2.$$

□

5 L^p - $L^{p'}$ Dissipation with $1 \leq p \leq 2$

Using the Riesz–Thorin theorem in Chapter 6 of [13] to interpolate the L^1 - L^∞ dissipation and the L^2 - L^2 dissipation, we obtain the L^p - $L^{p'}$ dissipation, where

$$1 \leq p \leq 2$$

and

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

given by the following theorem.

Theorem 5.1 *For all positive numbers ρ and all functions f in $L^p(\mathbb{H}^1)$, $1 \leq p \leq 2$,*

$$\|e^{-\rho \Delta_{\mathbb{H}^1}} f\|_{L^{p'}(\mathbb{H}^1)} \leq \frac{1}{2^{5/2} \pi^{3/2} \rho^{(p+2)/(2p)}} \|f\|_{L^p(\mathbb{H}^1)}.$$

Proof Let

$$\alpha_1 = 1, \beta_1 = 0, M_1 = \frac{1}{2^{5/2} \pi^{3/2} \rho^{3/2}}$$

and

$$\alpha_2 = \frac{1}{2}, \beta_2 = \frac{1}{2}, M_2 = \frac{1}{2^{5/2} \pi^{3/2} \rho}.$$

Let $\theta \in [0, 1]$ be such that

$$\frac{1}{p} = \alpha = (1 - \theta)\alpha_1 + \theta\alpha_2 = (1 - \theta) + \frac{\theta}{2} = 1 - \frac{\theta}{2}.$$

Then

$$\theta = \frac{2}{p'}$$

and

$$\beta = (1 - \theta)\beta_1 + \theta\beta_2 = \frac{1}{p'}.$$

So, by the Riesz–Thorin theorem, we get for all f in $L^p(\mathbb{H}^1)$,

$$\|e^{-\rho\Delta_{\mathbb{H}^1}}\|_{L^{p'}(\mathbb{H}^1)} \leq M_1^{1-\theta} M_2^\theta \|f\|_{L^p(\mathbb{H}^1)} = \frac{1}{2^{5/2}\pi^{3/2}\rho^{(p+2)/(2p)}} \|f\|_{L^p(\mathbb{H}^1)}.$$

□

Remark 5.2 It is obvious that the proof of Theorem 5.1 depends heavily on the assumption that $1 \leq p \leq 2$. It is interesting to compare this result with the Hausdorff–Young inequality for the Fourier transform on $\mathbb{R}^n$, which is only valid for $1 \leq p \leq 2$. Whether or not Theorem 5.1 is also true for $p > 2$ is worth further investigation.

6 The Quantum Wave

It is clear from the definition (1.1) that $\Delta_{\mathbb{H}^1}$ is a symmetric operator from $L^2(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ with dense domain given by the Schwartz space $\mathcal{S}$ on $\mathbb{R}^3$. Therefore it is closable and we denote its smallest closed extension again by $\Delta_{\mathbb{H}^1}$.

Proposition 6.1 *$\Delta_{\mathbb{H}^1}$ is closed and symmetric in $L^2(\mathbb{R}^3)$.*

Proof That $\Delta_{\mathbb{H}^1}$ is closed is obvious. The symmetry comes from a limiting argument. □

By Proposition 6.1, we see that Theorem X.1 on page 136 of the book [6] by Reed and Simon can be used to conclude that $\Delta_{\mathbb{H}^1}$ is self-adjoint if the resolvent set of $\Delta_{\mathbb{H}^1}$ contains a real number. From [1], we know that the spectrum of $\Delta_{\mathbb{H}^1}$ is precisely the interval $[0, \infty)$. Therefore $\Delta_{\mathbb{H}^1}$ is self-adjoint. So, we have the following theorem, which follows from Stone’s theorem given in, for instance, Section VIII.4 of the book [7] by Reed and Simon.

Theorem 6.2 *Let $\rho \in (-\infty, \infty)$. Then $e^{-i\rho\Delta_{\mathbb{H}^1}}$ is a unitary operator on $L^2(\mathbb{H}^1)$.*

In view of Theorem 6.2, we have the following result on the evolution of the quantum wave generated by the Laplacian on the Heisenberg group.

Remark 6.3 The partial differential equation in (1.3) is well known to be the Schrödinger equation driven by the Laplacian $\Delta_{\mathbb{H}^1}$ on the Heisenberg group $\mathbb{H}^1$. The function f in (1.3) is interpreted as the quantum state of the system at time $\rho = 0$. The solution $e^{-i\rho\Delta_{\mathbb{H}^1}}f$ can then be looked at as the quantum state or the wave function at time ρ . This wave function is probabilistic in nature and must be distinguished from the solution, known as the classical wave function, of the classical wave equation governed by the Laplacian $\Delta_{\mathbb{H}^1}$ on the Heisenberg group $\mathbb{H}^1$. We find it instructive to call the probabilistic wave function $e^{-i\rho\Delta_{\mathbb{H}^1}}f$ the quantum wave corresponding to the initial state f . See [5] and [10] for the basic notions in quantum mechanics.

Theorem 6.4 *For all $\rho \in (-\infty, \infty)$ and all functions $f \in L^2(\mathbb{R}^3)$,*

$$\|e^{-i\rho\Delta_{\mathbb{H}^1}}f\|_{L^2(\mathbb{R}^3)} = \|f\|_{L^2(\mathbb{R}^3)}.$$

Theorem 6.4 is often referred to as the law of conservation of energy.

Acknowledgment: This research has been partially supported by the Natural Sciences and Engineering Council of Canada.

Author Contributions: This is the confirmation from the corresponding author that the two listed authors are the only authors and each author has agreed to the content, equal contributions to the research, writing and checking on the content of this paper.

Declarations on Conflict of Interest: The authors declare no competing interests.

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