

Holomorphic One-Parameter Semigroups of Bounded Linear Operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, Generated by Strongly M -Elliptic Pseudo-Differential Operators

Yaodong Gao and M. W. Wong¹

Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada

E-Mail: yaodong@yorku.ca, mwwong@yorku.ca

Abstract For a M -elliptic pseudo-differential operator on $\mathbb{R}^n$, we give a sufficient condition on its symbol to ensure that it is the infinitesimal generator of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Key Words pseudo-differential operators, symbols in $M_{\rho, \Lambda}^m$, L^p -boundedness, M -ellipticity, strong M -ellipticity, holomorphic one-parameter semigroups

2020 Mathematics Subject Classification Primary: 47D06, 47G30

1 Introduction

Let $\Lambda \in C^\infty(\mathbb{R}^n)$ be a positive function such that there exist positive constants μ , C_1 and C_2 for which $\mu \geq 1$ and

$$C_1(1 + |\xi|)^\mu \leq \Lambda(\xi) \leq C_2(1 + |\xi|)^\mu, \quad \xi \in \mathbb{R}^n.$$

Suppose that for all multi-indices α and γ with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$, we can find a positive constant $C_{\alpha, \gamma}$ for which

$$|\xi^\gamma (\partial^{\alpha+\gamma} \Lambda)(\xi)| \leq C_{\alpha, \gamma} \Lambda(\xi)^{1 - \frac{1}{\mu} |\alpha|}, \quad \xi \in \mathbb{R}^n. \quad (1.1)$$

¹This research has been supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

For $m \in \mathbb{R}$ and $\rho = \frac{1}{\mu}$, $S_{\rho,\Lambda}^m$ is defined to be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β , there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{m-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

If

$$\Lambda(\xi) = \sqrt{1 + |\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

then $S_{\rho,\Lambda}^m$ is the same as the Hörmander class $S_{\rho,0}^m$ in [7]. The definitions adopted in the paper are in deference to the legacy of $1 \geq \rho > \delta \geq 0$ due to Hörmander. Let $\sigma \in S_{\rho,\Lambda}^m$. Then we define the pseudo-differential operator T_σ associated to the symbol σ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\varphi}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

for all φ in the Schwartz space $\mathcal{S}$, where the Fourier transform $\hat{\varphi}$ of φ is defined by

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^n.$$

For $m \in \mathbb{R}$ and $\rho = \frac{1}{\mu}$, we let $M_{\rho,\Lambda}^m$ be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices γ with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$,

$$\xi^\gamma (\partial_\xi^\gamma \sigma)(x, \xi) \in S_{\rho,\Lambda}^m.$$

When $\Lambda(\xi) = \sqrt{1 + |\xi|^2}$, $\xi \in \mathbb{R}^n$, $M_{\rho,\Lambda}^m$ can be traced back to the book [11] by Taylor.

Pseudo-differential operators with symbols in $M_{\rho,\Lambda}^m$ have been studied in [5, 6, 12]. Standard results such as asymptotic expansions, products, and formal adjoints similar to those for pseudo-differential operators with symbols given in the book by [13] are used freely.

Let $\sigma \in M_{\rho,\Lambda}^m$. Then it is easy to prove that $T_\sigma : \mathcal{S} \rightarrow \mathcal{S}$ is a continuous linear mapping. Hence we can use the formal adjoint to extend the definition of a pseudo-differential operator from the Schwartz space $\mathcal{S}$ to the space $\mathcal{S}'$ of all tempered distributions. To wit, let $\sigma \in M_{\rho,\Lambda}^m$. Then for all u in $\mathcal{S}'$, we define the linear functional $T_\sigma u$ on $\mathcal{S}$ by

$$(T_\sigma u)(\varphi) = u(\overline{T_\sigma^* \varphi}), \quad \varphi \in \mathcal{S}.$$

It is easy to see that T_σ maps $\mathcal{S}'$ into $\mathcal{S}'$ continuously. In fact, we have the following theorem.

Theorem 1.1 *Let $\sigma \in M_{\rho,\Lambda}^0$. Then for $1 < p < \infty$, $T_\sigma : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a bounded linear operator.*

A proof of Theorem 1.1 can be found in [5, 6]. To extend Theorem 1.4 to pseudo-differential operators with symbols in $M_{\rho,\Lambda}^m$, where m is an arbitrary real number, we need L^p Sobolev spaces of weight (ρ, Λ) . To this end, we first prove a proposition.

Proposition 1.2 $\Lambda \in M_{\rho,\Lambda}^1$.

Proof Let α be a multi-index. Then by (1.1), there exists a positive constant C_α such that

$$|(\partial^\alpha \Lambda)(\xi)| \leq C_\alpha \Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

Since $\rho = \frac{1}{\mu}$, it follows that

$$|(\partial^\alpha \Lambda)(\xi)| \leq C_\alpha \Lambda(\xi)^{1-\rho|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

Therefore

$$\Lambda \in S_{\rho,\Lambda}^1.$$

Now, let γ be a multi-index with $\gamma_j \in \{0, 1\}$, $j = 1, \dots, n$. Let α be a multi-index. Then

$$\begin{aligned} & |\partial^\alpha (\xi^\gamma (\partial^\gamma \Lambda)(\xi))| \\ &= \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta \xi^\gamma) (\partial^{\alpha-\beta+\gamma} \Lambda)(\xi) \right| \\ &= \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \binom{\gamma}{\beta} \beta! \xi^{\gamma-\beta} (\partial^{\alpha-\beta+\gamma} \Lambda)(\xi) \right| \end{aligned}$$

for all ξ in $\mathbb{R}^n$. Since the entries of $\gamma - \beta$ are in $\{0, 1\}$ and $\rho = \frac{1}{\mu}$, it follows that

$$\xi^\gamma (\partial^\gamma \Lambda) \in S_{\rho,\Lambda}^1.$$

This completes the proof. $\square$

Now, let $s \in \mathbb{R}$. Then we define $\Lambda(D)^s$ to be the Fourier multiplier given by

$$\Lambda(D)^s u = \mathcal{F}^{-1} \Lambda^s \mathcal{F} u, \quad u \in \mathcal{S}',$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier transform and the inverse Fourier transform.

For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the L^p -Sobolev space $H_\Lambda^{s,p}$ of weight (ρ, Λ) by

$$H_\Lambda^{s,p} = \{u \in \mathcal{S}' : \Lambda(D)^s u \in L^p(\mathbb{R}^n)\}.$$

Then $H_\Lambda^{s,p}$ is a Banach space in which the norm $\|\cdot\|_{s,p,\Lambda}$ is given by

$$\|u\|_{s,p,\Lambda} = \|\Lambda(D)^s u\|_{L^p(\mathbb{R}^n)}, \quad u \in H_\Lambda^{s,p}.$$

Details on these Sobolev spaces can be found in [12]. We mention in particular the following theorem.

Theorem 1.3 *Let $\sigma \in M_{\rho,\Lambda}^m$. Then for $-\infty < s < \infty$ and $1 < p < \infty$, $T_\sigma : H_\Lambda^{s,p} \rightarrow H_\Lambda^{s-m,p}$ is a bounded linear operator.*

A symbol σ in $M_{\rho,\Lambda}^m$ is said to be M -elliptic if there exist positive constants C and R such that

$$|\sigma(x, \xi)| \geq C \Lambda(\xi)^m, \quad |\xi| \geq R.$$

Theorem 1.4 *Let $\sigma \in M_{\rho,\Lambda}^m$ be M -elliptic. Then there exists a symbol τ in $M_{\rho,\Lambda}^{-m}$ such that*

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R and S are pseudo-differential operators with symbols in $\bigcap_{k \in \mathbb{R}} M_{\rho,\Lambda}^k$.

The pseudo-differential operator T_τ in the preceding theorem is known as a parametrix of the M -elliptic pseudo-differential operator T_σ .

Let $\sigma \in M_{\rho,\Lambda}^m$, $m > 0$. Then $T_\sigma : \mathcal{S} \rightarrow \mathcal{S}$ is a continuous linear mapping. So, we can consider T_σ as a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with

dense domain $\mathcal{S}$, where $1 < p < \infty$. It is closable and we denote the smallest closed extension by $T_{\sigma,0}$. See [12] for more information.

The aim of this paper is to prove that for a privileged elliptic symbol in $M_{\rho,\Lambda}^m$, $m > 0$, to be called a strongly M -elliptic symbol, the corresponding pseudo-differential operator generates a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. To this end, we give a deeper analysis in Section 2 of a class of symbols, motivated by Shubin [10] and first studied in [8], that relate to the fundamental class of strongly M -elliptic symbols. That strongly M -elliptic pseudo-differential operators generate holomorphic one-parameter semigroups of bounded linear operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, is proved in Section 3.

It should also be mentioned that the results in this paper generalize the results in [4] from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$. The techniques used in this paper, however, are drastically different from those in [4] in the sense that a version of Gårding's inequality, valid for $L^2(\mathbb{R}^n)$ and instrumental in [4], is no longer available for $L^p(\mathbb{R}^n)$ when $p \neq 2$.

2 Related Symbols and Strongly M -Elliptic Symbols

Let q be a fixed but arbitrary positive number and let Ω be an open subset of the complex plane $\mathbb{C}$. Then for any real number m , we define $S_{\rho,\Lambda,q,\Omega}^m$ to be the set of all functions σ on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega$ such that

$$\sigma(\cdot, \cdot, \lambda) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

for every $\lambda \in \Omega$ and for all multi-indices α and β , there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi, \lambda)| \leq C_{\alpha,\beta} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $\lambda \in \Omega$, where $\text{sgn} : \mathbb{R} \rightarrow \{0, \pm 1\}$ is given by

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

We denote by $M_{\rho, \Lambda, q, \Omega}^m$ the set of all functions σ in $S_{\rho, \Lambda, q, \Omega}^m$ such that for all multi-indices γ with $\gamma_j \in \{0, 1\}$, $1, 2, \dots, n$,

$$\xi^\gamma (D_\xi^\gamma \sigma)(x, \xi, \lambda) \in S_{\rho, \Lambda, q, \Omega}^m.$$

Let $\sigma \in M_{\rho, \Lambda, q, \Omega}^m$. Suppose that there exist positive constants C and R such that

$$|(\sigma(x, \xi, \lambda))| \geq C \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^m$$

for all $\lambda \in \Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$. Then we say that σ is $M_{\rho, \Lambda, q, \Omega}^m$ -elliptic.

A symbol σ in $M_{\rho, \Lambda}^m$ is said to be strongly M -elliptic if there exist positive constants C and R such that

$$\text{Re } \sigma(x, \xi) \geq C \Lambda(\xi)^m, \quad |\xi| \geq R.$$

By specializing Ω to open sectors with center at the origin of the complex plane $\mathbb{C}$, we have the following theorem.

Theorem 2.1 *Let σ be a strongly M -elliptic symbol in $M_{\rho, \Lambda}^m$, $m > 0$. Then there exists a number θ in $(0, \pi/2)$ such that $\sigma_{\Omega(\theta)}$ is $M_{\rho, \Lambda, m, \Omega(\theta)}^m$ -elliptic, where*

$$\Omega(\theta) = \{\lambda \in \mathbb{C} : \theta - \pi < \arg \lambda < \pi - \theta\}$$

and $\sigma_{\Omega(\theta)}$ is the function on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega(\theta)$ defined by

$$\sigma_{\Omega(\theta)}(x, \xi, \lambda) = \sigma(x, \xi) + \lambda$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$.

Proof We begin with the observation that $\sigma_{\Omega(\theta)} \in M_{\rho, \Lambda}^m$. Indeed, we first assume that $|\alpha| + |\beta| = 0$. Then there exists a positive constant C such that for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$,

$$\begin{aligned} |\sigma(x, \xi) + \lambda| &\leq C \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m + |\lambda| \\ &\leq C \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m + \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m \\ &\leq (C + 1) \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m. \end{aligned}$$

Now, for all multi-indices α and β with $|\alpha| \neq 0$ or $|\beta| \neq 0$, we can find a positive constant $C_{\alpha,\beta}$ such that for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$,

$$\begin{aligned} |(D_x^\alpha D_\xi^\beta(\sigma + \lambda))(x, \xi)| &= |(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \\ &\leq C_{\alpha,\beta} \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^{m-\rho|\beta|}. \end{aligned}$$

Now, we need to show there exists an angle θ and two positive constants C and R , which may depend on θ , such that

$$\begin{aligned} |\sigma(x, \xi) + \lambda| &\geq C \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m \\ &= C(\Lambda(\xi) + |\lambda|^{1/m})^m \end{aligned}$$

for all $\lambda \in \Omega(\theta)$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$. To this end, let c and C be positive numbers with $c < C$. Let $S_{c,C}$ be the circular segment given by

$$S_{c,C} = \{x \in \mathbb{C} : c \leq \operatorname{Re} z \leq |z| \leq C\}.$$

Let ϕ be the subtended angle of $S_{c,C}$. Then for all z in $S_{c,C}$ and all w in $\Omega(\theta)$,

$$\frac{|z + w|}{1 + |w|} \geq \frac{|z| - |w|}{1 + |w|} \geq \frac{c - |w|}{1 + |w|} \rightarrow c \quad (2.1)$$

as $|w| \rightarrow 0$ and

$$\frac{|z + w|}{1 + |w|} = \frac{\left| \frac{z}{|w|} + 1 \right|}{\frac{1}{|w|} + 1} \geq \frac{1 - \left| \frac{z}{|w|} \right|}{\frac{1}{|w|} + 1} \geq \frac{1 - \frac{C}{|w|}}{\frac{1}{|w|} + 1} \rightarrow 1 \quad (2.2)$$

as $|w| \rightarrow \infty$. Let $\varepsilon \in (0, \min(c, 1))$. Then by (2.1), there exists a number $\delta \in (0, c)$ such that

$$0 < |w| < \delta \Rightarrow \frac{|z + w|}{1 + |w|} \geq c - \varepsilon.$$

By (2.2), there exists a number $R \in (C, \infty)$ such that

$$|w| \geq R \Rightarrow \frac{|z + w|}{1 + |w|} \geq 1 - \varepsilon.$$

Now, let $A_{\delta,R}$ be the set given by

$$A_{\delta,R} = \{w \in \mathbb{C} : \delta \leq |w| \leq R\}.$$

Then the function $f : S_{c,C} \times A_{\delta,R} \rightarrow (0, \infty)$ defined by

$$f(z, w) = \frac{|z + w|}{1 + |w|}, \quad (z, w) \in S_{c,C} \times A_{\delta,R}$$

is a positive and continuous function on a compact set. So, there exists a positive constant M such that that

$$\frac{|z + w|}{1 + |w|} \geq M, \quad (z, w) \in S_{c,C} \times A_{\delta,R}.$$

Thus, we can find a positive constant K such that

$$|z + w| \geq K(1 + |w|), \quad z \in S_{c,C}, w \in \Omega(\theta). \quad (2.3)$$

Since there exist positive constants c and R such that

$$\operatorname{Re} \sigma(x, \xi) \geq C\Lambda(\xi)^m, \quad |\xi| \geq R,$$

it follows that $z = \frac{\sigma(x, \xi)}{\Lambda(\xi)^m}$, where $|\xi| \geq R$, satisfies

$$\operatorname{Re} z \geq c$$

and

$$|z| \leq C.$$

Let $w = \frac{\lambda}{\Lambda(\xi)^m}$, where $|\xi| \geq R$ and $\lambda \in \Omega(\theta)$. Then using (2.3), we get

$$|\sigma(x, \xi) + \lambda| \geq K(\Lambda(\xi)^m + |\lambda|),$$

which is the same as the existence of another positive constant K' for which

$$|\sigma(x, \xi) + \lambda| \geq K'(\Lambda(\xi) + \lambda^{1/m})^m$$

for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$ and all $\lambda \in \Omega(\theta)$. $\square$

Theorem 2.2 *Under the hypotheses of Theorem 2.1, there exists a symbol τ in $M_{\rho,\Lambda,m,\Omega(\theta_1)}^{-m}$ such that*

$$T_\tau T_{\sigma_{\Omega(\theta_1)}} = I + R,$$

where R is a pseudo-differential operator with symbol in $\bigcap_{k \in \mathbb{R}} M_{\rho,\Lambda,m,\Omega(\theta_1)}^k$

We need a characterization of elements in $M_{\rho,\Lambda,q,\Omega}^m$.

Proposition 2.3 *$\sigma \in M_{\rho,\Lambda,q,\Omega}^m$ if and only if for all multi-indices α, β and γ with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$, there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that*

$$|\xi^\gamma (D_x^\alpha D_\xi^{\beta+\gamma} \sigma)(x, \xi, \lambda)| \leq C_{\alpha,\beta,\gamma} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$.

Proof Let $\sigma \in M_{\rho,\Lambda,q,\Omega}^m$. Then $\sigma \in S_{\rho,\Lambda,q,\Omega}^m$. We have for all multi-indices γ with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$,

$$\xi^\gamma (\partial_\xi^\gamma \sigma)(x, \xi, \lambda) \in S_{\rho,\Lambda,q,\Omega}^m.$$

Therefore for all multi-indices α and β , there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$\begin{aligned} & |(D_x^\alpha D_\xi^\beta (\xi^\gamma (D_\xi^\gamma \sigma)))(x, \xi, \lambda)| \\ & \leq \sum_{\beta' \leq \beta, \beta' \leq \gamma} \binom{\beta}{\beta'} |(D_\xi^{\beta'} \xi^\gamma) (D_x^\alpha D_\xi^{\beta-\beta'+\gamma} \sigma)(x, \xi, \lambda)| \\ & = \sum_{\beta' \leq \beta, \beta' \leq \gamma} \binom{\beta}{\beta'} \binom{\gamma}{\beta'} \beta'! |\xi^{\gamma-\beta'} (D_x^\alpha D_\xi^{\beta+(\gamma-\beta')} \sigma)(x, \xi, \lambda)| \\ & \leq C_{\alpha,\beta,\gamma} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}. \end{aligned} \quad (2.4)$$

Conversely, suppose that for all multi-indices α, β and γ with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$, there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\xi^\gamma (D_x^\alpha D_\xi^{\beta+\gamma} \sigma)(x, \xi, \lambda)| \leq C_{\alpha,\beta,\gamma} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$. Let γ be the zero multi-index. Then there exists a positive constant $C_{\alpha, \beta}$ such that

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi, \lambda)| \leq C_{\alpha, \beta} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$. Therefore $\sigma \in S_{\rho, \Lambda, q, \Omega}^m$. Now, for all multi-indices α, β and γ with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$, the derivation of (2.4) shows that $\xi^\gamma (D_\xi^\gamma \sigma)(x, \xi, \gamma) \in S_{\rho, \Lambda, q, \Omega}^m$. Thus, $\sigma \in M_{\rho, \Lambda, q, \Omega}$. $\square$

To prove Theorem 2.2, we note that the same proof of Theorem 4.5.1 in [8], which is built on the foundation laid out in the book [13], gives a parametrix T_τ of $T_{\Omega(\theta_1)}$, where $\tau \in S_{\rho, \Lambda, m, \omega(\theta_1)}^{-m}$. So, it remains to prove that $\tau \in M_{\rho, \Lambda, m, \Omega(\theta_1)}^{-m}$. To do this, let α, β and γ be multi-indices with $\gamma_j \in \{0, 1\}$, $j = 1, 2, \dots, n$. Then there exists a positive constant $C_{\alpha, \beta, \gamma}$ such that

$$\begin{aligned} & |(D_x^\alpha D_\xi^\beta (\xi^\gamma (D_\xi^\gamma \tau)))(x, \xi, \lambda)| \\ & \leq C_{\alpha, \beta, \gamma} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} \right)^m \Lambda(\xi)^{-m-\rho(|\beta|)} \\ & \leq C_{\alpha, \beta, \gamma} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} \right)^m \Lambda(\xi)^{-m-\rho|\beta|-\rho|\gamma-\beta'|+|\gamma-\beta'|/\mu} \\ & = C_{\alpha, \beta, \gamma} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} \right)^m \Lambda(\xi)^{-m-\rho|\beta|}. \end{aligned}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$. Here we have used the assumption that $\rho = \frac{1}{\mu}$. Therefore

$$\tau \in M_{\rho, \Lambda, m, \Omega(\theta_1)}^{-m}.$$

3 Holomorphic One-Parameter Semigroups

To make the paper more self-contained, let us first recall what a holomorphic one-parameter semigroup of bounded linear operators on a complex Banach space is. Good references are the books [2], [3] and [9], among others.

Definition 3.1 Let Ω be the sector in the complex plane given by

$$\Omega = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2, \text{ where } \varphi_1 < 0 < \varphi_2\}.$$

Let $\{T(z) : z \in \Omega\}$ be a family of bounded linear operators on a complex Banach space X such that

- the mapping $\mathbb{C} \ni z \mapsto T(z) \in B(X)$, where $B(X)$ is the Banach space of all bounded linear operators on X , is holomorphic,
- $T(z_1 + z_2) = T(z_1)T(z_2)$, $z_1, z_2 \in \Omega$,
- for all $x \in X$,

$$\lim_{z \rightarrow 0} T(z)x = x.$$

Then we call $\{T(z) \in B(X) : z \in \Omega\}$ a holomorphic one-parameter semigroup of bounded linear operators on the complex Banach space X .

We begin with a result on infinitesimal generators of holomorphic one-parameter semigroups of bounded linear operators on complex Banach spaces in [9].

Theorem 3.2 *Let A be a closed linear operator densely defined on a complex Banach space X such that $\Omega(\theta) \cup \{0\}$ is a subset of the resolvent set $\rho(A)$ of A for some θ in $(0, \pi/2)$ and there exists a positive constant M for which*

$$\|(A - \lambda I)^{-1}\| \leq \frac{M}{|\lambda|}$$

for all λ in $\rho(A)$ with $\theta - \pi < \arg \lambda < \pi - \theta$ and $\lambda \neq 0$, where $\|\cdot\|$ is the norm in the Banach algebra of all bounded linear operators on X and I is the identity operator on X . Then A is the infinitesimal generator of a holomorphic one-parameter semigroup $\{T(z) : |\arg z| < \theta\}$ of bounded linear operators on X .

The following variant of Theorem 3.2 is needed.

Theorem 3.3 *Let A be a closed linear operator densely defined on a complex Banach space X such that*

$$\Omega(\theta) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq R\} \subset \rho(A)$$

for some θ in $(0, \pi/2)$ and some positive number R . In addition, we suppose that

$$\|(A - \lambda I)^{-1}\| \leq \frac{M}{|\lambda|}$$

for all λ in $\rho(A)$ with

$$\theta - \pi < \arg \lambda < \pi - \theta$$

and $|\lambda| \geq R$. Then A is the infinitesimal generator of a holomorphic one-parameter semigroup $\{T(z) : |\arg z| < \theta\}$ of bounded linear operators on X .

Proof Let $\tilde{A} = A - RI$. Then for all $\lambda \in \Omega(\theta) \cup \{0\}$,

$$\tilde{A} - \lambda I = (A - RI) - \lambda I = A - (\lambda + R)I,$$

which is bijective. So, $\lambda \in \rho(\tilde{A})$. Therefore

$$\Omega(\theta) \cup \{0\} \subset \rho(\tilde{A}).$$

Since

$$\tilde{A} - \lambda I = A - (\lambda + R)I,$$

we see that

$$\|(\tilde{A} - \lambda I)^{-1}\| \leq \frac{M}{|\lambda + R|} \leq \frac{M}{|\lambda|}.$$

By Theorem 3.2, $\tilde{A}$ is the infinitesimal generator of a holomorphic one-parameter semigroup $\{\tilde{T}(z) : |\arg z| < \theta\}$ of bounded linear operators on X . For all $z \in \mathbb{C}$ with $|\arg z| < \theta$, let

$$T(z) = e^{Rz} \tilde{T}(z).$$

Then A is the infinitesimal generator of a holomorphic one-parameter semigroup $\{T(z) : |\arg z| < \theta\}$ of bounded linear operators on X . $\square$

The next theorem is an estimate that captures the essence of the Agmon–Douglis–Nirenberg estimates in [1] for pseudo-differential operators on $\mathbb{R}^n$.

Theorem 3.4 *Let $\sigma \in M_{\rho, \Lambda}^m$, $m > 0$, be a strongly M -elliptic symbol. Then we can find positive constants K_1 , R_1 and θ_1 with $\theta_1 \in (0, \pi/2)$ such that*

$$K_1 \|u\|_{m, p, \Lambda} \leq \|(T_{\sigma, 0} + \lambda I)u\|_{0, p, \Lambda} + \|u\|_{0, p, \Lambda}, \quad u \in H_{\Lambda}^{m, p},$$

for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| > R_1$, where

$$\Omega(\theta_1) = \{\lambda \in \mathbb{C} : \theta_1 - \pi < \arg \lambda < \pi - \theta_1\},$$

I is the identity operator on $L^p(\mathbb{R}^n)$ and $1 < p < \infty$.

Proof Let $\sigma_{\Omega(\theta_1)}$ be the function on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega(\theta_1)$ defined by

$$\sigma_{\Omega(\theta_1)}(x, \xi, \lambda) = \sigma(x, \xi) + \lambda, \quad (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \Omega(\theta_1).$$

Since σ is strongly M -elliptic, it follows from Theorem 2.1 that $\sigma_{\Omega(\theta_1)}$ is $M_{\rho, \Lambda, m, \Omega(\theta_1)}^m$ -elliptic. By Theorem 2.2, there exists a symbol τ in $M_{\rho, \Lambda, m, \Omega(\theta_1)}^{-m}$ such that

$$T_\tau T_{\sigma_{\Omega(\theta_1)}} = I + R,$$

where R is a pseudo-differential operator with symbol in $\bigcap_{k \in \mathbb{R}} M_{\rho, \Lambda, m, \Omega(\theta_1)}^k$. For all multi-indices α and β , there exists a positive constant $C_{\alpha, \beta}$ such that

$$|(D_x^\alpha D_\xi^\beta \tau)(x, \xi, \lambda)| \leq C_{\alpha, \beta} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} \right)^m \Lambda(\xi)^{-m - \rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta_1)$. Since

$$\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} < 1, \quad \xi \in \mathbb{R}^n, \lambda \in \Omega(\theta_1),$$

it follows that

$$|(D_x^\alpha D_\xi^\beta \tau)(x, \xi, \lambda)| \leq C_{\alpha, \beta} \Lambda(\xi)^{-m - \rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta_1)$. By the L^p -boundedness of pseudo-differential operators with symbols in $M_{\rho, \Lambda}^m$ given by Theorem 1.3, there exist two positive constants C' and C'' such that

$$\begin{aligned} \|u\|_{m, p, \Lambda} &= \|(T_\tau T_{\sigma_{\Omega(\theta_1)}} + R)u\|_{m, p, \Lambda} \\ &\leq \|T_\tau T_{\sigma + \lambda} u\|_{m, p, \Lambda} + \|Ru\|_{m, p, \Lambda} \\ &\leq C' \|T_{\sigma + \lambda} u\|_{0, p, \Lambda} + C'' \|u\|_{0, p, \Lambda}, \quad u \in H_\Lambda^{m, p}, \end{aligned}$$

for all $\lambda \in \Omega(\theta_1)$. Let $Q = \max(C', C'')$. Then

$$\|u\|_{m, p, \Lambda} \leq Q(\|T_{\sigma + \lambda} u\|_{0, p, \Lambda} + \|u\|_{0, p, \Lambda}), \quad u \in H_\Lambda^{m, p}.$$

Let $K = 1/Q$. Then

$$K\|u\|_{m, p, \Lambda} \leq \|T_{\sigma + \lambda} u\|_{0, p, \Lambda} + \|u\|_{0, p, \Lambda}, \quad u \in H_\Lambda^{m, p}. \quad (3.1)$$

Since $\sigma \in M_{\rho, \Lambda}^m$ is M -elliptic, it follows from the Agmon–Douglis–Nirenberg estimates in [1] that we can find two positive constants C_1 and C_2 for which

$$C_1 \|u\|_{m, p, \Lambda} \leq \|T_{\sigma, 0} u\|_{0, p, \Lambda} + \|u\|_{0, p, \Lambda} \leq C_2 \|u\|_{m, p, \Lambda}, \quad u \in H_\Lambda^{m, p}. \quad (3.2)$$

By (3.1) and (3.2), we get

$$\|T_{\sigma,0}u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \leq \frac{C_2}{K} \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \frac{C_2}{K} \|u\|_{0,p,\Lambda}.$$

Therefore

$$\|(T_{\sigma,0} + \lambda I)u - \lambda u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \leq \frac{C_2}{K} \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \frac{C_2}{K} \|u\|_{0,p,\Lambda},$$

which implies that

$$\begin{aligned} & |\lambda| \|u\|_{0,p,\Lambda} - \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \\ & \leq \frac{C_2}{K} \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \frac{C_2}{K} \|u\|_{0,p,\Lambda}. \end{aligned}$$

Thus,

$$\left(|\lambda| + 1 - \frac{C_2}{K}\right) \|u\|_{0,p,\Lambda} \leq \left(1 + \frac{C_2}{K}\right) \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda}.$$

Now, let $\omega = \frac{C_2}{K} - 1$ and $L = 1 + \frac{C_2}{K}$. Then

$$\frac{|\lambda| - \omega}{L} \|u\|_{0,p,\Lambda} \leq \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda}.$$

Let $\tilde{R}$ be a positive number such that

$$1 - \frac{\omega}{\tilde{R}} > 0.$$

Then for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| \geq \tilde{R}$, we have

$$\tilde{C} |\lambda| \|u\|_{0,p,\Lambda} \leq \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda}, \quad u \in H_{\Lambda}^{m,p}, \quad (3.3)$$

for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| \geq \tilde{R}$ if we pick $\tilde{C}$ to be any positive number such that

$$\tilde{C} < \frac{1 - \frac{\omega}{\tilde{R}}}{L}.$$

□

The same proof of Theorem 3.4 gives a similar estimate for the adjoint $T_{\sigma,0}^t$ of the pseudo-differential operator $T_{\sigma,0}$.

Theorem 3.5 *Let $\sigma \in M_{\rho, \Lambda}^m$, $m > 0$, be a strongly M -elliptic symbol. Then we can find positive constants K_2 , R_2 and θ_2 with $\theta_2 \in (0, \pi/2)$ such that*

$$K_2 \|u\|_{m, p', \Lambda} \leq \|(T_{\sigma, 0}^t + \bar{\lambda}I)u\|_{0, p', \Lambda} + \|u\|_{0, p', \Lambda}, \quad u \in H_{\Lambda}^{m, p'},$$

for all $\lambda \in \Omega(\theta_2)$ with $|\lambda| > R_2$, where

$$\Omega(\theta_2) = \{\lambda \in \mathbb{C} : \theta_2 - \pi < \arg \lambda < \pi - \theta_2\},$$

I is the identity operator on $L^{p'}(\mathbb{R}^n)$ and

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We can now state and prove the main result of this paper.

Theorem 3.6 *Let $\sigma \in M_{\rho, \Lambda}^m$, $m > 0$, be a strongly M -elliptic symbol. Then the pseudo-differential operator $-T_{\sigma, 0}$ is the infinitesimal generator of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

Proof By Theorem 3.4, $T_{\sigma, 0} + \lambda I$ is injective and has closed range for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| > R_1$. By Theorem 3.5, $T_{\sigma, 0}^t + \bar{\lambda}I$ is injective and has closed range for all $\lambda \in \Omega(\theta_2)$ with $|\lambda| > R_2$. Let

$$\theta = \min(\theta_1, \theta_2),$$

$$R = \max(R_1, R_2),$$

and

$$\Omega(\theta) = \{\lambda \in \mathbb{C} : \theta - \pi < \arg \lambda < \pi - \theta\}.$$

Let $\lambda \in \Omega(\theta)$ be such that $|\lambda| > R$. Then for any function v in $H_{\Lambda}^{0, p'}$ annihilating the range of $T_{\sigma, 0} + \lambda I$, we have

$$v((T_{\sigma, 0} + \lambda I)u) = 0, \quad u \in H_{\Lambda}^{m, p}.$$

Therefore

$$u((T_{\sigma, 0}^t + \bar{\lambda}I)v) = 0, \quad u \in H_{\Lambda}^{m, p}.$$

So, $(T_{\sigma,0}^t + \bar{\lambda}I)v = 0$. Using the injectivity of $T_{\sigma,0}^t + \bar{\lambda}I$, we see that $v = 0$. Thus, the range of $T_{\sigma,0} + \lambda I$ is dense in $H_{\Lambda}^{0,p}$. Since the range of $T_{\sigma,0} + \lambda I$ is closed in $H_{\Lambda}^{0,p}$, it follows that the range of $T_{\sigma,0} + \lambda I$ is equal to $H_{\Lambda}^{0,p}$. Therefore $T_{\sigma,0} + \lambda I$ is surjective. This implies that $-T_{\sigma,0} - \lambda I$ is also surjective. Hence for all $\lambda \in \Omega(\theta)$ with $|\lambda| > R$,

$$\lambda \in \rho(-T_{\sigma,0})$$

and by (3.3),

$$\|(-T_{\sigma,0} - \lambda I)^{-1}\| \leq \frac{M}{|\lambda|},$$

where $\|\cdot\|$ is the norm in the Banach algebra of all bounded linear operators on $L^p(\mathbb{R}^n)$ and

$$M = \frac{1}{\bar{C}}.$$

Therefore by Theorem 3.3, $-T_{\sigma,0}$ is the infinitesimal generator of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$. $\square$

Confirmation This is the confirmation from the corresponding author that the two listed authors are the only authors and each author has agreed to the content, equal contributions to the research, writing and checking on the content of this paper.

Declaration There are no competing interests on the content of this paper.

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