Heat Semigroups and Inverses of Twisted Laplacians on Nonisotropic Heisenberg Groups with Multi-Dimensional Center[1]

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Abstract: A formula for the heat semigroups generated by the twisted Laplacians in terms of Weyl transforms is given. L^p–L^q estimates for the heat semigroups are also given. The twisted Laplacians on Heisenberg groups with multi-dimensional center are shown to be globally hypoelliptic in the Schwartz space and in the Gelfand–Shilov spaces using the Green functions of the twisted Laplacians. Global regularity in a scale of Sobolev spaces for these twisted Laplacians are presented as well.

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1 Introduction

The aim of this paper is to look at some aspects of analysis and partial differential equations on a class of nonisotropic Heisenberg groups with multi-dimensional center. We begin with a description of the Laplacians on the nonisotropic Heisenberg groups with multi-dimensional center.

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Let $B_1, B_2, \ldots, B_m$ be $n \times n$ orthogonal matrices with real entries such that

$$B_j^{-1} B_k = -B_k^{-1} B_j$$

for all $j, k = 1, 2, \ldots, m$ with $j \neq k$. Then we define the nonisotropic Heisenberg group $\mathbb{G}$ with multi-dimensional center to be the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ equipped with the binary operation $\cdot$ given by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{2}[z, w] \right)$$

for all (z, t) and (w, s) in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, where $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $t, s \in \mathbb{R}^m$ and $[z, w] \in \mathbb{R}^m$ is given by

$$[z, w]_j = u \cdot B_j y - x \cdot B_j v, \quad j = 1, 2, \ldots, m.$$

The center Z of the nonisotropic Heisenberg group $\mathbb{G}$ with multi-dimensional center is m-dimensional and is given by

$$Z = \{(0, 0, t) : t \in \mathbb{R}^m\}.$$

The following proposition on the dimension of a nonisotropic Heisenberg group and the dimension of its center can be found in [8].

Proposition 1.1 *Let $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ be the underlying manifold of a nonisotropic Heisenberg group. Then*

$$m^2 \leq n.$$

Nonisotropic Heisenberg groups with multi-dimensional center are special cases of H-type groups in [5, 6, 7]. If $m = 1$ and $B_1 = -I_n$, where I_n is the $n \times n$ identity matrix, then we get back the ordinary n-dimensional Heisenberg group $\mathbb{H}^n$. In the book [15], for the sake of simplifying the notation and making the presentation transparent, we have chosen to study in detail the one-dimensional Heisenberg group $\mathbb{H}^1$.

Let $\mathfrak{g}$ be the Lie algebra of all left-invariant vector fields on $\mathbb{G}$. For $j = 1, 2, \ldots, n$, let $\gamma_{1,j} : \mathbb{R} \to \mathbb{G}$ and $\gamma_{2,j} : \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{1,j}(s) = (se_j, 0, 0)$$

and

$$\gamma_{2,j}(s) = (0, se_j, 0)$$

for all $s \in \mathbb{R}$, where e_j is the standard unit vector in $\mathbb{R}^n$ with 1 in the j^{th} position. For $k = 1, 2, \ldots, m$, let $\gamma_{3,k} : \mathbb{R} \to \mathbb{G}$ be the curve in $\mathbb{G}$ given by

$$\gamma_{3,k}(s) = (0, 0, se_k)$$

for all $s \in \mathbb{R}$, where e_k is the standard unit vector in $\mathbb{R}^m$ with 1 in the k^{th} position. For $j = 1, 2, \ldots, n$, we define the left-invariant vector fields X_j and Y_j by

$$\begin{aligned}
& (X_j f)(x, y, t) \\
= \; & \frac{d}{ds} f((x, y, t) \cdot \gamma_{1,j}(s)) \Big|_{s=0} \\
= \; & \frac{d}{ds} f\left(x + se_j, y, \left(t_1 + \frac{1}{2}(B_1 y, se_j), \ldots, t_m + \frac{1}{2}(B_m y, se_j)\right)\right)\Big|_{s=0} \\
= \; & \frac{\partial f}{\partial x_j}(x, y, t) + \frac{1}{2}\sum_{k=1}^{m}(B_k y, e_j)\frac{\partial f}{\partial t_k}(x, y, t)
\end{aligned}$$

and

$$\begin{aligned}
& (Y_j f)(x, y, t) \\
= \; & \frac{d}{ds} f((x, y, t) \cdot \gamma_{2,j}(s)) \Big|_{s=0} \\
= \; & \frac{d}{ds} f\left(x, y + se_j, \left(t_1 - \frac{1}{2}(x, sB_1 e_j), \ldots, t_m - \frac{1}{2}(x, sB_m e_j)\right)\right)\Big|_{s=0} \\
= \; & \frac{\partial f}{\partial y_j}(x, y, t) - \frac{1}{2}\sum_{k=1}^{m}(x, B_k e_j)\frac{\partial f}{\partial t_k}(x, y, t)
\end{aligned}$$

for all $(x, y, t) \in \mathbb{G}$ and all $f \in C^\infty(\mathbb{G})$. For $k = 1, 2, \ldots, m$, we define the vector field T_k by

$$\begin{aligned}
& (T_k f)(x, y, t) \\
= \; & \frac{d}{ds} f((x, y, t) \cdot \gamma_{3,k}(s)) \Big|_{s=0} \\
= \; & \frac{d}{ds} f(x, y, t + se_k) \Big|_{s=0} \\
= \; & \frac{\partial f}{\partial t_k}(x, y, t)
\end{aligned}$$

for all $(x, y, t) \in \mathbb{G}$ and all $f \in C^\infty(\mathbb{G})$. We can easily check that

$$[X_j, Y_l] = -\sum_{k=1}^{m} (B_k)_{jl} T_k, \quad j, l = 1, 2, \ldots, n,$$

and the other commutators are zero.

Theorem 1.2 *The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is generated by*

$$\{X_j, Y_l, [X_j, Y_l] : j, l = 1, 2, \ldots, n\}.$$

The sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ is defined by

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

Explicitly,

$$\begin{aligned} \mathcal{L} &= -\Delta_x - \Delta_y - \frac{1}{4}(|x|^2 + |y|^2)\Delta_t \\ &\quad + \sum_{j=1}^{n} \sum_{k=1}^{m} \left[-(B_k y, e_j) \frac{\partial}{\partial x_j} + (x, B_k e_j) \frac{\partial}{\partial y_j} \right] \frac{\partial}{\partial t_k}. \end{aligned}$$

Let $\mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}$. Then by taking the inverse Fourier transform of the sub-Laplacian with respect to t, we get parametrized twisted Laplacians L^λ, $\lambda \in \mathbb{R}^{m*}$, given by

$$L^\lambda = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2)|\lambda|^2 - i \sum_{j=1}^{n} \left\{ -(B_\lambda y, e_j) \frac{\partial}{\partial x_j} + (x, B_\lambda e_j) \frac{\partial}{\partial y_j} \right\}, \tag{1.1}$$

where

$$B_\lambda = \sum_{j=1}^{m} \lambda_j B_j.$$

To recapitulate, the first four sections in this paper provide a recall of the nonisotropic Heisenberg group with multi-dimensional center, a family of twisted Laplacians on it parametrized by $\lambda \in \mathbb{R}^{m*}$ in the center, the heat kernels and the Green functions of these twisted Laplacians. We first give the $L^p - L^q$ estimates of the heat semigroup, also known as the strongly continuous one-parameter semigroup, generated by L^λ for all $\lambda \in \mathbb{R}^{m*}$. We also prove that for all $\lambda \in \mathbb{R}^{m*}$, L^λ is globally hypoelliptic in the Schwartz space $\mathcal{S}$ and in the Gelfand–Shilov spaces S^μ_ν, where μ and ν are positive numbers with $\mu + \nu \geq 1$. Global hypoellipticity of differential operators can be found in [2]. In [3], global hypoellipticity, renamed global regularity, of second order twisted differential operators is further developed. Analogs of these resuts for the ordinary Heisenberg group with one-dimensional center can be found in, respectively, [4] and [13]. In addition, we construct a scale of Sobolev spaces to measure the global regularity of L^λ. The results on the heat semigroup are given in Sections 5-8. The results on global hypoellipticity are given in Sections 9 and 10. Essential self-adjointness and global regularity are given in, respectively, Sections 11 and 12.

2 Spectral Analysis of λ-Twisted Laplacians

For $k = 0, 1, 2, \ldots$, the Hermite function e_k of order k on $\mathbb{R}$ is defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define the function e_α on $\mathbb{R}^n$ by

$$e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n}.$$

For all $\lambda \in \mathbb{R}^{m*}$ and all multi-indices α and β in $(\mathbb{N} \cup \{0\})^n$, we define the special Hermite function $e^\lambda_{\alpha,\beta}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$e^\lambda_{\alpha,\beta}(q,p) = |\lambda|^{n/2} V^\lambda(e_\alpha, e_\beta) \left(\frac{q}{\sqrt{|\lambda|}}, \sqrt{|\lambda|} p\right), \quad q, p \in \mathbb{R}^n,$$

where

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(B\lambda^t q)\cdot y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy, \quad q, p \in \mathbb{R}^n, \tag{2.1}$$

for all f and g in $\mathcal{S}$. The matrix B_λ^t is the transpose of B_λ. In fact, $e_{\alpha,\beta}^\lambda$ is given by

$$e_{\alpha,\beta}^\lambda(q,p) = V^\lambda(e_\alpha^\lambda, e_\beta^\lambda)(q, \sqrt{|\lambda|}p), \quad q,p \in \mathbb{R}^n,$$

where

$$e_\alpha^\lambda(x) = |\lambda|^{n/4} e_\alpha(\sqrt{|\lambda|}x), \quad x \in \mathbb{R}^n.$$

Theorem 2.1 *$\{e_{\alpha,\beta}^\lambda : \alpha,\beta \in (\mathbb{N}\cup\{0\})^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.*

The following theorem gives the spectral analysis of the λ-twisted Laplacian for all $\lambda \in \mathbb{R}^{m*}$.

Theorem 2.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all multi-indices α and β in $(\mathbb{N}\cup\{0\})^n$,*

$$L^\lambda e_{\alpha,\beta}^\lambda = |\lambda|^n(2|\beta|+n)e_{\alpha,\beta}^\lambda.$$

All definitions and results in this section can be found in [9].

3 λ-Weyl Transforms

We have defined the λ-Fourier–Wigner transform $V^\lambda(f,g)$ of f and g in $\mathcal{S}$ by (2.1). In fact,

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(B_\lambda^t q)\cdot x} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}\,dx, \quad q,p \in \mathbb{R}^n.$$

It is easy to see that the λ-Fourier–Wigner transform is related to the ordinary Fourier–Wigner transform by

$$V^\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q, p), \quad q,p \in \mathbb{R}^n.$$

Note that

$$V^\lambda(f,g)(q,-p) = \overline{V^\lambda(g,f)}(q,p), \quad q,p \in \mathbb{R}^n.$$

Now, we define the λ-Wigner transform $W^\lambda(f,g)$ of f and g in $L^2(\mathbb{R}^n)$ to be the Fourier transform of $V^\lambda(f,g)$. In fact, the λ-Wigner transform has the form

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(\frac{B_\lambda^t x}{|\lambda|^2}+\frac{p}{2}\right)\overline{g\left(\frac{B_\lambda^t x}{|\lambda|^2}-\frac{p}{2}\right)}\,dp \tag{3.1}$$

for all x and ξ in $\mathbb{R}^n$, and and it is related to the ordinary Wigner transform by

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n} W(f,g)\left(\frac{B_\lambda^t x}{|\lambda|^2}, \xi\right)$$

for all x, ξ in $\mathbb{R}^n$. Moreover,

$$W^\lambda(f,g) = \overline{W^\lambda(g,f)}, \quad f,g \in L^2(\mathbb{R}^n).$$

Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then we define the λ-Weyl transform $W_\sigma^\lambda f$ of f corresponding to the symbol σ by

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi) W^\lambda(f,g)(x,\xi)\, dx\, d\xi, \tag{3.2}$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Therefore using Parseval's identity, we have

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) V^\lambda(f,g)(q,p)\, dq\, dp.$$

Hence, formally, we can write

$$\left(W_\sigma^\lambda f\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) \left(\pi_\lambda(q,p) f\right)(x)\, dq\, dp, \quad x \in \mathbb{R}^n.$$

Proposition 3.1 *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is given by*

$$W_\sigma^\lambda = W_{\sigma_\lambda},$$

where W_{σ_λ} is the ordinary Weyl transform corresponding to the symbol σ_λ given by

$$\sigma_\lambda(x,\xi) = \sigma(B_\lambda x, \xi), \quad x, \xi \in \mathbb{R}^n.$$

Let F and G be functions in $L^2(\mathbb{R}^{2n})$. Then the λ-twisted convolution $F *_\lambda G$ of F and G is the function on $\mathbb{R}^{2n}$ defined by

$$(F *_\lambda G)(z) = \int_{\mathbb{R}^{2n}} F(z-w) G(w) e^{\frac{i}{2}\lambda \cdot [z,w]}\, dw, \quad z \in \mathbb{R}^{2n}, \tag{3.3}$$

provided that the integral exists.

Theorem 3.2 *Let σ and τ be in $L^2(\mathbb{R}^{2n})$. Then*

$$W_\sigma^\lambda W_\tau^\lambda = W_\omega^\lambda,$$

where $\omega \in L^2(\mathbb{R}^{2n})$ and $\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_\lambda \hat{\tau})$.

We have the following Moyal identity for the λ-Wigner transform and the λ-Fourier–Wigner transform.

Proposition 3.3 *For all f_1, f_2, g_1, g_2 in $L^2(\mathbb{R}^n)$,*

$$\big(W^\lambda(f_1, g_1), W^\lambda(f_2, g_2)\big) = |\lambda|^{-n}\, (f_1, f_2)\, \overline{(g_1, g_2)}$$

and

$$\big(V^\lambda(f_1, g_1), V^\lambda(f_2, g_2)\big) = |\lambda|^{-n}\, (f_1, f_2)\, \overline{(g_1, g_2)}.$$

4 Heat Kernels and Green Functions of λ-Twisted Laplacians

We recall in this section the heat kernel and the Green function of the λ-twisted Laplacian L^λ, $\lambda \in \mathbb{R}^{m*}$, given in [9]. We first give the heat kernel of the λ-twisted Laplacian L^λ, which is the kernel of the integral operator $e^{-\tau L^\lambda}$ for $\tau > 0$. The twisted convolution defined in (3.3) is the key ingredient in the following theorem.

Theorem 4.1 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and all $\tau > 0$,*

$$e^{-\tau L^\lambda} f = k_\tau^\lambda *_{-\lambda} f,$$

where

$$k_\tau^\lambda(z) = (2\pi)^{-n} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n \tau)]^n} e^{-\frac{1}{4}|\lambda|\,|z|^2 \coth(|\lambda|^n \tau)}$$

for all $z \in \mathbb{R}^{2n}$.

As a corollary, the heat kernel κ_τ^λ of the λ-twisted Laplacian L^λ for $\lambda \in \mathbb{R}^{m*}$ is given by

$$\begin{aligned} \kappa_\tau^\lambda(z, w) &= k_\tau^\lambda(z - w) e^{-\frac{i}{2}\lambda\cdot[z,w]} \\ &= (2\pi)^{-n} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n \tau)]^n} e^{-\frac{1}{4}|\lambda|\,|z-w|^2 \coth(\tau|\lambda|^n)} e^{-\frac{i}{2}\lambda\cdot[z,w]} \end{aligned} \quad (4.1)$$

for all z and w in $\mathbb{R}^{2n}$.

The Green function G^λ of L^λ is the kernel of the inverse $(L^\lambda)^{-1}$. The Green function G^λ is related to the heat kernel κ_τ^λ of L^λ by

$$G^\lambda(z, w) = \int_0^\infty \kappa_\tau^\lambda(z, w)\, d\tau, \quad z, w \in \mathbb{R}^{2n}.$$

Let

$$g^\lambda(z) = \int_0^\infty k_\tau^\lambda(z)\, d\tau, \quad z \in \mathbb{R}^{2n}.$$

Then the Green function G^λ of L^λ is given by

$$G^\lambda(z,w) = e^{-\frac{i}{2}\lambda\cdot[z,w]} g^\lambda(z-w)$$

for all z and w in $\mathbb{R}^{2n}$. An explicit formula for g^λ is given in the following theorem.

Theorem 4.2 *For all* $z \in \mathbb{R}^{2n}$,

$$g^\lambda(z) = \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \frac{\Gamma(n/2)}{(\sqrt{|\lambda|}|z|)^{n-1}} K_{(n-1)/2}\left(\frac{1}{4}|\lambda|\,|z|^2\right),$$

where $K_{(n-1)/2}$ *is the modified Bessel function of order* $(n-1)/2$ *given by*

$$K_{(n-1)/2}(x) = \int_0^\infty e^{-x\cosh\delta}\cosh((n-1)\delta/2)\, d\delta, \quad x > 0.$$

5 Heat Semigroups Generated by λ-Twisted Laplacians on $\mathbb{G}$

A formula for the heat semigroup $e^{-\tau L^\lambda}$, $\tau > 0$, on $\mathbb{G}$ is given in the following theorem. It is an analog of the formula for the heat semigroup generated by the twisted Laplacian in the one-dimensional Heisenberg group given in [14].

Theorem 5.1 *Let* $f \in L^2(\mathbb{R}^{2n})$. *Then for all* $\lambda \in \mathbb{R}^{m*}$ *with* $|\lambda| = 1$ *and* $\tau > 0$,

$$e^{-\tau L^\lambda} f = (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)} V^\lambda(W_{\hat{f}}^\lambda e_\beta, e_\beta).$$

Proof Let $f \in \mathcal{S}$. Then for $\tau > 0$, we have

$$e^{-\tau L^\lambda} f = \sum_\beta \sum_\alpha e^{-\tau|\lambda|^n(2|\beta|+n)}(f, e_{\alpha,\beta}^\lambda) e_{\alpha,\beta}^\lambda, \tag{5.1}$$

where the series is convergent in $L^2(\mathbb{R}^n)$. Now, using the λ-Wigner transform and the Plancherel theorem,

$$\begin{aligned}(f, e^\lambda_{\alpha,\beta}) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{2n}} f(z)\overline{V^\lambda(e_\alpha, e_\beta)(z)}\,dz \\ &= \int_{\mathbb{R}^{2n}} \hat{f}(\zeta)\overline{V^\lambda(e_\alpha, e_\beta)^\wedge(\zeta)}\,d\zeta \\ &= \int_{\mathbb{R}^{2n}} \hat{f}(\zeta)\overline{W^\lambda(e_\alpha, e_\beta)(\zeta)}\,d\zeta \\ &= (2\pi)^{n/2}(W^\lambda_{\hat{f}} e_\beta, e_\alpha). \end{aligned} \tag{5.2}$$

Similarly, for all $g \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$(e^\lambda_{\alpha,\beta}, g) = \overline{(g, e^\lambda_{\alpha,\beta})} = (2\pi)^{n/2}\overline{(W^\lambda_{\hat{g}} e_\beta, e_\alpha)} = (2\pi)^{n/2}(e_\alpha, W^\lambda_{\hat{g}} e_\beta). \tag{5.3}$$

So, by (5.1), (5.2) and (5.3),

$$\begin{aligned}(e^{-\tau L^\lambda} f, g) &= (2\pi)^n \sum_\beta \sum_\alpha e^{-\tau(2|\beta|+n)}(W^\lambda_{\hat{f}} e_\beta, e_\alpha)(e_\alpha, W^\lambda_{\hat{g}} e_\beta) \\ &= (2\pi)^n \sum_\beta e^{-\tau(2|\beta|+n)} \sum_\alpha (W^\lambda_{\hat{f}} e_\beta, e_\alpha)(e_\alpha, W^\lambda_{\hat{g}} e_\beta) \\ &= (2\pi)^n \sum_\beta e^{-\tau(2|\beta|+n)}(W^\lambda_{\hat{f}} e_\beta, W^\lambda_{\hat{g}} e_\beta) \end{aligned} \tag{5.4}$$

for all $\tau > 0$. Using the definition of the λ-Weyl transform and Plancherel's theorem,

$$\begin{aligned}(W^\lambda_{\hat{f}} e_\beta, W^\lambda_{\hat{g}} e_\beta) &= (2\pi)^{-n/2}\overline{\int_{\mathbb{R}^{2n}} \hat{g}(z) W^\lambda(e_\beta, W^\lambda_{\hat{f}} e_\beta)(z)\,dz} \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^{2n}} W^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)(z)\overline{\hat{g}(z)}\,dz \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^{2n}} V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)(z)\overline{g(z)}\,dz \end{aligned} \tag{5.5}$$

for all multi-indices β. By (5.4) and (5.5),

$$\begin{aligned}(e^{-\tau L^\lambda} f, g) &= (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)}(V^\lambda(W^\lambda_{\hat{f}} e^\beta, e_\beta), g) \\ &= (2\pi)^{n/2}\left(\sum_\beta e^{-\tau(2|\beta|+n)} V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta), g\right) \end{aligned}$$

for all f and g in $\mathcal{S}(\mathbb{R}^n)$ and all $\tau > 0$. Thus,

$$e^{-\tau L^\lambda} f = (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)} V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $\tau > 0$. □

6 An L^p–L^2 Estimate

We begin with the following improvement of Theorem 11.1 in [13].

Theorem 6.1 *Let $\sigma \in L^p(\mathbb{R}^{2n})$, $1 \le p \le 2$. Then for $\lambda \in \mathbb{R}^{m*}$, the λ-Weyl transform $W^\lambda_{\hat{\sigma}}$, originally defined on $\mathcal{S}(\mathbb{R}^n)$, can be extended to a unique bounded linear operator on $L^2(\mathbb{R}^n)$. Moreover,*

$$\|W^\lambda_{\hat{\sigma}}\|_* \le (2\pi)^{-n/2}(2/|\lambda|)^{n(1-(2/p'))}\|\sigma\|_{L^p(\mathbb{R}^{2n})},$$

where $\| \, \|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$.

Proof Let $\sigma \in L^2(\mathbb{R}^{2n})$. Then for all f and g in $L^2(\mathbb{R}^n)$, we get by (3.2), the Schwarz inequality and the Plancherel theorem,

$$\begin{aligned} |(W^\lambda_{\hat{\sigma}} f, g)_{L^2(\mathbb{R}^n)}| &\le (2\pi)^{-n/2}\|\hat{\sigma}\|_{L^2(\mathbb{R}^{2n})}\|W^\lambda(f,g)\|_{L^2(\mathbb{R}^{2n})} \\ &= (2\pi)^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^{2n})}\|W^\lambda(f,g)\|_{L^2(\mathbb{R}^{2n})}. \end{aligned}$$

By Moyal's identity in Proposition 3.3,

$$\|W^\lambda(f,g)\|_{L^2(\mathbb{R}^{2n})} = \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

Therefore

$$|(W^\lambda_{\hat{\sigma}} f, g)_{L^2(\mathbb{R}^n)}| \le (2\pi)^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}, \quad f, g \in L^2(\mathbb{R}^n).$$

So,

$$\|W^\lambda_{\hat{\sigma}} f\|_{L^2(\mathbb{R}^n)} \le (2\pi)^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n).$$

Now, let $\sigma \in L^1(\mathbb{R}^{2n})$. Then for all f and g in $L^2(\mathbb{R}^n)$, we get by (3.2) and Hölder's inequality,

$$|(W^\lambda_{\hat{\sigma}} f, g)_{L^2(\mathbb{R}^{2n})}| \le (2\pi)^{-n/2}\|\hat{\sigma}\|_{L^1(\mathbb{R}^{2n})}\|V^\lambda(f,g)\|_{L^\infty(\mathbb{R}^{2n})}.$$

By (3.1) and the Schwarz inequality,

$$\begin{aligned}
& \|W^\lambda(f,g)\|_{L^\infty(\mathbb{R}^{2n})} \\
\leq\ & (2\pi)^{-n/2}|\lambda|^{-n}\left[\int_{\mathbb{R}^n}\left|f\left(\frac{B_\lambda^t x}{|\lambda|^2}+\frac{p}{2}\right)\right|^2 dp\right]^{\frac{1}{2}}\left[\int_{\mathbb{R}^n}\left|g\left(\frac{B_\lambda^t x}{|\lambda|^2}-\frac{p}{2}\right)\right|^2 dp\right]^{\frac{1}{2}} \\
=\ & (2\pi)^{-n/2}|\lambda|^{-n}2^n\|\sigma\|_{L^1(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Let $\sigma \in L^p(\mathbb{R}^{2n})$. Then by the Riesz–Thorin theorem, we get for all f in $L^2(\mathbb{R}^n)$,

$$\begin{aligned}
& \|W_{\hat{\sigma}}^\lambda f\|_{L^2(\mathbb{R}^n)} \\
\leq\ & [(2\pi)^{-n/2}]^{2/p'}[(2\pi)^{-n/2}2^n|\lambda|^{-n}]^{1-(2/p')}\|\sigma\|_{L^p(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)} \\
=\ & (2\pi)^{-n/2}(2/|\lambda|)^{n(1-(2/p'))}\|\sigma\|_{L^p(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}. \qquad (6.1)
\end{aligned}$$

By (6.1), the proof is complete. □

Theorem 6.2 *For $\tau > 0$, the heat semigroup $e^{-\tau L^\lambda}$ with $|\lambda| = 1$, initially defined on $\mathcal{S}(\mathbb{R}^{2n})$, can be extended to a unique bounded linear operator from $L^p(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$, which we again denote by $e^{-\tau L^\lambda}$, and*

$$\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^{2n})} \leq 2^{n(1-(2/p'))}\frac{1}{[2\sinh\,\tau]^n}\|f\|_{L^p(\mathbb{R}^{2n})}$$

for all $f \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq 2$.

Proof By Theorem 5.1, Minkowski's inequality and the Moyal identity for the λ-Fourier–Wigner transform, we get for all $f \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\begin{aligned}
\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^{2n})} &\leq (2\pi)^{n/2}\sum_\beta e^{-\tau(2|\beta|+n)}\|V^\lambda(W_{\hat{f}}^\lambda e_\beta, e_\beta))\|_{L^2(\mathbb{R}^{2n})} \\
&= (2\pi)^{n/2}\sum_\beta e^{-\tau(2|\beta|+n)}\|W_{\hat{f}}^\lambda e_\beta\|_{L^2(\mathbb{R}^n)}\|e_\beta\|_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{n/2}|\lambda|^n\sum_\beta e^{-\tau|\lambda|^n(2|\beta|+n)}\|W_{\hat{f}}^\lambda e_\beta\|_{L^2(\mathbb{R}^n)} \qquad (6.2)
\end{aligned}$$

for $\tau > 0$. So, by (6.2) and Theorem 6.1, we get for $\tau > 0$,

$$\begin{aligned}
\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^{2n})} &\leq \sum_\beta e^{-\tau(2|\beta|+n)}\|f\|_{L^p(\mathbb{R}^{2n})} \\
&= 2^{n(1-(2/p'))}\frac{1}{[2\sinh\,(|\lambda|^n\tau)]^n}\|f\|_{L^p(\mathbb{R}^{2n})}.
\end{aligned}$$

$\square$

7 L^p–L^∞ Estimates, $1 \le p \le \infty$

We begin with the following theorem.

Theorem 7.1 *Let $\lambda \in \mathbb{R}^{m*}$, Then for all $\tau > 0$ and $1 \le p \le \infty$, $e^{-\tau L^\lambda}$: $L^p(\mathbb{R}^{2n}) \to L^\infty(\mathbb{R}^{2n})$ is a bounded linear operator. More precisely,*

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^{2n})} \le (2\pi)^{-n/p} \frac{|\lambda|^{n/p}}{[\sinh(|\lambda|^n \tau)]^{n/p}} \frac{1}{[\cosh(|\lambda|^n \tau)]^{n/p'}} \|f\|_{L^p(\mathbb{R}^{2n})}$$

for all $f \in L^p(\mathbb{R}^{2n})$.

Proof Using the formula (4.1) for the heat kernel κ_τ^λ of L^λ,

$$|\kappa_\tau^\lambda(z,w)| \le a_\tau$$

for all z and w in $\mathbb{C}^n$ and $\tau > 0$, where

$$a_\tau = (2\pi)^{-n} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n \tau)]^n}.$$

So, for all $f \in L^1(\mathbb{R}^{2n})$,

$$|(e^{-\tau L^\lambda} f)(z)| \le \int_{\mathbb{C}^n} |\kappa(z,w)|\,|f(w)|\,dw = a_\tau \|f\|_{L^1(\mathbb{R}^{2n})}$$

for all $z \in \mathbb{C}^n$. Therefore

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^{2n})} \le a_\tau \|f\|_{L^1(\mathbb{R}^{2n})}.$$

Now, for all $f \in L^\infty(\mathbb{R}^{2n})$,

$$\begin{aligned} |(e^{-\tau L^\lambda} f)(z)| &\le \|f\|_{L^\infty(\mathbb{R}^{2n})} \int_{\mathbb{C}^n} |\kappa_\tau(z,w)|\,dw \\ &\le a_\tau \int_{\mathbb{C}^n} e^{-\frac{1}{4}|\lambda|\,|w|^2 \coth(\tau|\lambda|^n)} dw\, \|f\|_{L^\infty(\mathbb{R}^{2n})} \end{aligned}$$

for all $z \in \mathbb{C}^n$. Thus,

$$\begin{aligned}\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^{2n})} &\leq a_\tau \frac{(4\pi)^n}{|\lambda|^n[\coth(|\lambda|^n\tau)]^n} \|f\|_{L^\infty(\mathbb{R}^{2n})} \\ &= \frac{1}{[\cosh(|\lambda|^n\tau)]^n} \|f\|_{L^\infty(\mathbb{R}^{2n})}.\end{aligned}$$

Using the Riesz–Thorin Theorem, we have

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^{2n})} \leq (2\pi)^{-n/p} \frac{|\lambda|^{n/p}}{[\sinh(|\lambda|^n\tau)]^{n/p}} \frac{1}{[\cosh(|\lambda|^n\tau)]^{n/p'}} \|f\|_{L^p(\mathbb{R}^{2n})}.$$

□

8 L^p–L^q Estimates, $1 \leq p \leq 2$, $2 \leq q \leq \infty$

Using Theorem 6.2 and Theorem 7.1, we have the following theorem.

Theorem 8.1 *For $\tau > 0$, the heat semigroup $e^{-\tau L^\lambda}$ with $|\lambda| = 1$, initially defined on $\mathcal{S}(\mathbb{R}^n)$, can be extended to a bounded linear operator from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$, which we again denote by $e^{-\tau L^\lambda}$ and*

$$\|e^{-\tau L^\lambda} f\|_{L^q(\mathbb{R}^{2n})} \leq \frac{2^{n(1-(2/p'))(2/q)}}{[\sinh \tau]^{(2n/q)+(n/p)(1-(2/q))}[\cosh \tau]^{(n/p')(1-(2/q)}} \|f\|_{L^p(\mathbb{R}^{2n})}$$

for all $f \in L^p(\mathbb{R}^{2n})$.

9 Global Hypoellipticity in the Schwartz Space

The Green function is now used to prove that the λ-twisted Laplacian L^λ with $\lambda \in \mathbb{R}^{m*}$ is globally hypoelliptic.

We need an estimate of the modified Bessel function K_ν of order ν, where $\nu > 0$.

Lemma 9.1 *Let $\nu > 0$. Then for every positive number η with $\eta \geq \nu$, there exists a positive constant C_η such that*

$$|K_\nu(x)| \leq C_\eta x^{-\eta}, \quad x > 0.$$

Proof Let η be a positive number such that $\eta \geq \nu$. Using the asymptotic behavior of $K_\nu(x)$ for large x in [10], we know that

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$$

as $x \to \infty$. So, there exists a positive constant C'_η such that for sufficiently large x, say, $x \geq R'$,

$$K_\nu(x) \leq \sqrt{\frac{\pi}{2x}} e^{-x} \leq C'_\eta x^{-((1/2)+\eta)} < C'_\eta x^{-\eta}.$$

Using the asymptotic behavior of $K_\nu(x)$ for small x in [10], we have

$$K_\nu(x) \sim 2^{\nu-1}\Gamma(\nu) x^{-\nu}$$

as $x \to 0+$. Then there exists a positive constant C''_η such that

$$K_\nu(x) \leq C''_\eta x^{-\eta}$$

for sufficiently small and positive values of x, say, $x \leq R''$. Since $\frac{K_\nu(x)}{x^{-\eta}}$ is continuous on $(0, \infty)$, it follows that there exists a positive constant C'''_η for which

$$K_\nu(x) \leq C'''_\eta x^{-\eta}, \quad x \in [R'', R'],$$

and the lemma is proved with $C_\eta = \max(C'_\eta, C''_\eta, C'''_\eta)$. □

We also need the following estimate.

Lemma 9.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all multi-indices γ on $\mathbb{R}^{2n}$,*

$$\left|\partial_z^\gamma \left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| \leq \left(\frac{1}{2}m|\lambda|\right)^{|\gamma|} |w|^{|\gamma|}, \quad z, w \in \mathbb{R}^{2n}.$$

Proof Writing $z = x + iy$ and $w = u + iv$, where x, y, u and v are in $\mathbb{R}^n$, we have

$$[z, w]_j = u \cdot B_j y - x \cdot B_j v, \quad j = 1, 2, \ldots, m.$$

Then for $j = 1, 2, \ldots, m$,

$$[z, w]_j = \sum_{l=1}^n (B_j^t u)_l y_l - \sum_{l=1}^n (B_j v)_l x_l$$

and hence

$$e^{-\frac{i}{2}\lambda\cdot[z,w]} = e^{-\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j v)_l x_l}.$$

We also write

$$\partial_z^\gamma = \partial_x^\theta \partial_y^\phi,$$

where θ and ϕ are multi-indices on $\mathbb{R}^n$ with $|\theta+\phi| = |\gamma|$. For $k = 1, 2, \ldots, n$,

$$\begin{aligned} & \partial_{y_k}\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j v)_l x_l} \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right) \end{aligned}$$

and hence

$$\begin{aligned} & \partial_{y_k}^{\phi_k}\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j v)_l x_l} \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right)^{\phi_k}. \end{aligned}$$

So,

$$\begin{aligned} & \partial_y^\phi\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^l (B_j^t u)_l y_j} e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j v)_l x_l} \prod_{k=1}^n \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right)^{\phi_k}. \end{aligned}$$

Differentiating the preceding equation with respect to x to the order θ, we obtain

$$\begin{aligned} & \partial_z^\gamma\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & \partial_x^\theta \partial_y^\phi\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j v)_l x_l} \\ & \left[\prod_{k=1}^n \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right)^{\phi_k}\right]\left[\prod_{k=1}^n \left(\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j v)_k\right)^{\theta_k}\right]. \end{aligned}$$

Therefore

$$\begin{aligned}&\left|\partial_z^\gamma\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right|\\ =\ &\left|\prod_{k=1}^n\left(-\frac{i}{2}\sum_{j=1}^m\lambda_j(B_j^t u)_k\right)^{\phi_k}\right|\left|\prod_{k=1}^n\left(\frac{i}{2}\sum_{j=1}^m\lambda_j(B_j v)_k\right)^{\theta_k}\right|.\end{aligned}$$

If we let $\|B_j\|$ denote the operator norm of B_j for $j=1,2,\ldots,m$, then

$$\begin{aligned}\left|\prod_{k=1}^n\left(-\frac{i}{2}\sum_{j=1}^m\lambda_j(B_j^t u)_k\right)^{\phi_k}\right| &\leq \prod_{k=1}^n\left(\frac{1}{2}\sum_{j=1}^m|\lambda_j||B_j^t u|\right)^{\phi_k}\\ &\leq \prod_{k=1}^n\left(\frac{1}{2}|\lambda|\left(\sum_{j=1}^m\|B_j\|\right)|u|\right)^{\phi_k}\\ &= \left(\frac{1}{2}|\lambda|\right)^{|\phi|}\left(\sum_{j=1}^m\|B_j\|\right)^{|\phi|}|u|^{|\phi|}\\ &\leq \left(\frac{1}{2}|\lambda|\right)^{|\phi|}\left(\sum_{j=1}^m\|B_j\|\right)^{|\phi|}|w|^{|\phi|}.\end{aligned}$$

Since B_j is an orthogonal matrix for $j=1,2,\ldots,m$, it follows that

$$\|B_j\|=1,\quad j=1,2,\ldots,m,$$

and hence

$$\left|\prod_{k=1}^n\left(-\frac{i}{2}\sum_{j=1}^m\lambda_j(B_j^t u)_k\right)^{\phi_k}\right|\leq\left(\frac{1}{2}m|\lambda|\right)^{|\phi|}|w|^{|\phi|}.$$

Similarly,

$$\left|\prod_{k=1}^n\left(\frac{i}{2}\sum_{j=1}^m\lambda_j(B_j v)_k\right)^{\theta_k}\right|\leq\left(\frac{1}{2}m|\lambda|\right)^{|\theta|}|w|^{|\theta|}.$$

Thus,

$$\left|\partial_z^\gamma\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right|\leq\left(\frac{1}{2}m|\lambda|\right)^{|\gamma|}|w|^{|\theta|}$$

and the proof is complete. □

We can now give the global hypoellipticity of the λ-twisted Laplacian L^λ with $\lambda \in \mathbb{R}^{m*}$ in the Schwartz space.

Theorem 9.3 *Let $\lambda \in \mathbb{R}^{m*}$. Then the λ-twisted Laplacian L^λ is globally hypoelliptic in the sense that*

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), L^\lambda u \in \mathcal{S}(\mathbb{R}^{2n}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{2n}),$$

where $\mathcal{S}'(\mathbb{R}^{2n})$ is the space of all tempered distributions on $\mathbb{R}^{2n}$.

Proof Let $f = L^\lambda u$. Then for all $z \in \mathbb{R}^{2n}$,

$$\begin{aligned} u(z) &= ((L^\lambda)^{-1} f)(z) \\ &= \int_{\mathbb{R}^{2n}} g^\lambda(w) f(z-w) e^{-\frac{i}{2}\lambda\cdot[z,w]} dw, \end{aligned}$$

where

$$g^\lambda(z) = \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \frac{\Gamma(n/2)}{(\sqrt{|\lambda|}|z|)^{n-1}} K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right).$$

Let β be any multi-index. Then for all $z \in \mathbb{R}^{2n}$,

$$(\partial^\beta u)(z) = \int_{\mathbb{R}^{2n}} g^\lambda(w) \partial_z^\beta \left(f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) dw.$$

To justify the interchange of differentiation and integration, we write for all $z \in \mathbb{R}^{2n}$,

$$\int_{\mathbb{R}^{2n}} |g^\lambda(w)| \left|\partial_z^\beta \left(f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| dw = I_1(z) + I_2(z),$$

where

$$I_1(z) = \int_{|w|\le 1} |g^\lambda(w)| \left|\partial_z^\beta \left(f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| dw$$

and

$$I_2(z) = \int_{\mathbb{R}^{2n}} |g^\lambda(w)| \left|\partial_z^\beta \left(f(z-w)e^{-\frac{1}{2}i\lambda\cdot[z,w]}\right)\right| dw.$$

Using the hypothesis that $f \in \mathcal{S}(\mathbb{R}^n)$, the definition of $[z,w]$, the formula of Leibniz to the effect that

$$\partial_z^\beta \left(f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} (\partial^{\beta-\gamma} f)(z-w) \partial^\gamma \left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)$$

and Lemma 9.2, we get

$$\sup_{z\in\mathbb{R}^{2n}} |I_1(z)| < \infty$$

and

$$\sup_{z\in\mathbb{R}^{2n}} |I_2(z)| < \infty.$$

Now, let α and β be arbitrary multi-indices with $\alpha \neq 0$. Then for all $z \in \mathbb{R}^{2n}$,

$$|z^\alpha(\partial^\beta u)(z)| \leq 2^{|\alpha|}(J_1(z) + J_2(z)),$$

where

$$J_1(z) = \int_{\mathbb{R}^{2n}} |w|^{|\alpha|}|g^\lambda(w)| \left|\partial_z^\beta \left(f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| dw$$

and

$$J_2(z) = \int_{\mathbb{R}^{2n}} |z-w|^{|\alpha|}|g^\lambda(w)| \left|\partial_z^\beta \left(f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| dw.$$

As in the case when $\alpha = 0$,

$$\sup_{z\in\mathbb{R}^{2n}} |J_1(z)| < \infty.$$

By breaking $\mathbb{R}^{2n}$ into $|w| \leq 1$ and $|w| \geq 1$, and using the same argument as in the case when $\alpha = 0$, we see that

$$\sup_{z\in\mathbb{R}^{2n}} |J_2(z)| < \infty,$$

and the proof is complete. □

10 Global Hypoellipticity in Gelfand–Shilov Spaces

Let μ and ν be positive real numbers such that $\mu+\nu \geq 1$. Then the Gelfand–Shilov space $S_\nu^\mu(\mathbb{R}^n)$ is defined to be the set of all functions φ in $C^\infty(\mathbb{R}^n)$ for which there exists a positive constant C such that for all multi-indices α and β,

$$|x^\alpha(\partial^\beta\varphi)(x)| \leq C^{|\alpha|+|\beta|+1}(\alpha!)^\nu(\beta!)^\mu, \quad x \in \mathbb{R}^n.$$

It can be shown that a function φ is in $S_\nu^\mu(\mathbb{R}^n)$ if and only if there exist positive constants C and ε such that for all muti-indices α,

$$|(\partial^\alpha \varphi)(x)| \le C^{|\alpha|+1} (\alpha!)^\mu e^{-\varepsilon |x|^{1/\nu}}, \quad x \in \mathbb{R}^n.$$

This characterization tells us that a function in a Gelfand–Shilov space has exponential decay at infinity. Moreover, a function φ is in the Gelfand–Shilov space $S_\nu^\mu(\mathbb{R}^n)$ if and only if there exist positive constants C and ε such that

$$|\varphi(x)| \le C e^{-\varepsilon |x|^{1/\nu}}, \quad x \in \mathbb{R}^n,$$

and

$$|\hat{\varphi}(\xi)| \le C e^{-\varepsilon} |\xi|^{1/\mu}, \quad \xi \in \mathbb{R}^n.$$

It is worth pointing out that the Gelfand–Shilov space $S_1^1(\mathbb{R}^n)$ is the same as the test space F for Fourier hyperfunctions. In fact, F is the set of all functions φ in $C^\infty(\mathbb{R}^n)$ for which there exist positive constants C, ε and δ such that for all multi-indices α,

$$|(\partial^\alpha)(x)| \le C \delta^{|\alpha|} \alpha! e^{-\varepsilon |x|}, \quad x \in \mathbb{R}^n.$$

We have the following theorem on the global hypoellipticity of the twisted Laplacian L^λ in Gelfand–Shilov spaces.

Theorem 10.1 *Let μ and ν be positive real numbers with $\mu + \nu \ge 1$. Then*

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), L^\lambda u \in S_\nu^\mu(\mathbb{R}^{2n}) \Rightarrow u \in S_\nu^\mu(\mathbb{R}^{2n}).$$

Proof Let $f \in S_\nu^\mu(\mathbb{R}^{2n})$. Then there exists a positive constant C such that for all multi-indices α and β,

$$|z^\alpha (\partial^\beta f)(z)| \le C^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad z \in \mathbb{C}^n. \tag{10.1}$$

As in the proof of Theorem 9.3, we need to estimate $I_1(z)$. To do this, we use the inequality (10.1), the definition of $[z, w]$ and the Leibniz formula to obtain a positive constant C_1 such that

$$I_1(z) \le C_1^{|\beta|+1} (\beta!)^\mu \int_{|w|\le 1} |g^\lambda(w)| \, dw.$$

By Lemma 9.2, we see that there exists a positive constant C_2 such that

$$I_1(z) \le C_2^{|\beta|+1} (\beta!)^\mu, \quad z \in \mathbb{C}^n.$$

Similarly, there exists a positive constant C_4 such that

$$I_2(z) \leq C_4^{|\beta|+1}(\beta!)^{\mu}, \quad z \in \mathbb{C}^n.$$

Then as in the proof of Theorem 9.3 again, we need to estimate $J_1(z)$ and $J_2(z)$. Using the same argument as in the case when $\alpha = 0$, we obtain a positive constants C_5 for which

$$J_1(z) \leq C_5^{|\alpha|+|\beta|+1}(\alpha!)^{\mu}(\beta!)^{\mu}, \quad z \in \mathbb{C}^n.$$

Using the Leibniz formula and Lemma 9.2, we get a positive constant C_6 such that

$$J_2(z) \leq C_6^{|\alpha|+|\beta|+1}(\alpha!)^{\nu}(\beta!)^{\mu} \int_{\mathbb{C}^n} |w|^{|\beta|}|g^{\lambda}(w)|\, dw, \quad z \in \mathbb{C}.$$

By breaking $\mathbb{C}^n$ into $|w| \leq 1$ and $|w| \geq 1$ and using Lemma 9.2, the proof is complete. □

11 Essential Self-Adjointness

Let $\lambda \in \mathbb{R}^{m*}$. Then using the explicit formula for the λ-twisted Laplacian L^{λ} given in (1.1), it can be checked easily that L^{λ} is a symmetric operator from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with dense domain $\mathcal{S}$. So, L^{λ} is closable and we denote the closure by L_0^{λ}.

Proposition 11.1 *Let $\lambda \in \mathbb{R}^{m*}$. Then L_0 is closed and symmetric.*

Proof We only need to prove that L_0^{λ} is symmetric. Let u and v be functions in the domain $\mathcal{D}(L_0^{\lambda})$ of L_0^{λ}. Then we can find sequences $\{\varphi_l\}_{l=1}^{\infty}$ and $\{\psi_l\}_{l=1}^{\infty}$ in $\mathcal{S}$ such that

$$\varphi_l \to u,$$
$$L^{\lambda}\varphi_l \to L_0^{\lambda}u,$$
$$\psi_l \to v$$

and

$$L^{\lambda}\psi_l \to L_0^{\lambda}v$$

in $L^2(\mathbb{R}^{2n})$ as $l \to \infty$. So, using the symmetry of L^λ as a linear operator from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with domain $\mathcal{S}$,

$$(L_0^\lambda u, v) = \lim_{l\to\infty}(L^\lambda \varphi_l, \psi_l) = \lim_{l\to\infty}(\varphi_l, L^\lambda \psi) = (u, L_0^\lambda v).$$

Therefore L_0^λ is symmetric. □

For all $\lambda \in \mathbb{R}^{m*}$, let $\Sigma(L_0^\lambda)$ be the spectrum of L_0^λ. Then we have the following theorem.

Theorem 11.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then*

$$\Sigma(L_0^\lambda) = \{|\lambda|^n(2|\beta| + n) : \beta \in (\mathbb{N} \cup \{0\})^n\}.$$

Moreover, for every $\beta \in (\mathbb{N} \cup \{0\})^n$, the number $|\lambda|^n(2|\beta| + n)$ is an eigenvalue of L_0^λ with infinite multiplicity.

Proof It follows from Theorem 2.2 that every number $|\lambda|^n(2|\beta| + n)$ with $\beta \in \mathbb{N} \cup \{0\}$ is an eigenvalue of L_0^λ with infinite multiplicity and hence is an element of $\Sigma(L_0^\lambda)$. Now, let $\mu \in \mathbb{C}$. Suppose that

$$\mu \neq |\lambda|^n(2|\beta| + n)$$

for all $\beta \in (\mathbb{N} \cup \{0\})^n$. If we can prove that the range $R(L_0^\lambda - \mu I)$ of $L_0^\lambda - \mu I$ is dense in $L^2(\mathbb{R}^{2n})$, where I is the identity operator on $L^2(\mathbb{R}^{2n})$, and there exists a positive constant C such that

$$\|(L_0^\lambda - \mu I)u\|_{L^2(\mathbb{R}^{2n})} \geq C\|u\|_{L^2(\mathbb{R}^{2n})}, \quad u \in \mathcal{D}(L_0^\lambda),$$

then μ lies in the resolvent set $\rho(L_0^\lambda)$ and the proof is then complete. Let M be the subspace of $L^2(\mathbb{R}^{2n})$ consisting of all finite linear combinations of elements in $\{e_{\alpha,\beta}^\lambda : \alpha, \beta \in (\mathbb{N} \cup \{0\})^n\}$. Then by Theorem 2.1, M is dense in $L^2(\mathbb{R}^{2n})$. Let $f \in M$. Then we can write

$$f = \sum_{|\alpha|\leq N_1} \sum_{|\beta|\leq N_2} a_{\alpha,\beta} e_{\alpha,\beta}^\lambda,$$

where N_1 and N_2 are positive integers and

$$a_{\alpha,\beta} \in \mathbb{C}, \quad |\alpha| \leq N_1, |\beta| \leq N_2.$$

Let

$$u = \sum_{|\alpha|\le N_1, |\beta|\le N_2} \frac{a_{\alpha,\beta}}{|\lambda|^n(2|\beta|+n)-\mu} e^\lambda_{\alpha,\beta}.$$

Let $C_\mu = \inf_{\beta\in(\mathbb{N}\cup\{0\})^n} ||\lambda|^n(2|\beta|+n)-\mu|$. Since

$$\mu \neq |\lambda|^n(2|\beta|+n)$$

for all $\beta \in (\mathbb{N}\cup\{0\})^n$, it follows that $C_\mu > 0$. Therefore $u \in \mathcal{S}$. Furthermore,

$$\begin{aligned}
(L_0^\lambda - \mu I)u &= (L^\lambda - \mu I)u = \sum_{|\alpha|\le N_1}\sum_{|\beta|\le N_2} \frac{a_{\alpha,\beta}}{|\lambda|^n(2|\beta|+n)-\mu} L^\lambda e^\lambda_{\alpha,\beta} \\
&= \sum_{|\alpha|\le N_1}\sum_{|\beta|\le N_2} a_{\alpha,\beta} e^\lambda_{\alpha,\beta} = f.
\end{aligned}$$

Therefore $f \in R(L_0^\lambda - \mu I)$. So, $M \subseteq R(L_0^\lambda - \mu I)$. This proves that $R(L_0^\lambda - \mu I)$ is dense in $L^2(\mathbb{R}^{2n})$. Let $u \in \mathcal{D}(L_0^\lambda)$. Then using the symmetry of L^λ, Theorem 2.1 and Parseval's identity,

$$\begin{aligned}
\|(L_0^\lambda - \mu I)u)\|^2_{L^2(\mathbb{R}^{2n})} &= \left\| \sum_\alpha \sum_\beta ((L_0^\lambda - \mu I)u, e^\lambda_{\alpha,\beta}) e^\lambda_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^{2n})} \\
&= \left\| \sum_\alpha \sum_\beta (u, ((L_0^\lambda)^* - \overline{\mu} I) e^\lambda_{\alpha,\beta}) e^\lambda_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^{2n})} \\
&= \left\| \sum_\alpha \sum_\beta (u, (L^\lambda - \overline{\mu} I) e^\lambda_{\alpha,\beta}) e^\lambda_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^{2n})} \\
&= \left\| \sum_\alpha \sum_\beta (u, (|\lambda|^n(2|\beta|+n) - \overline{\mu}) e^\lambda_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^{2n})} \\
&= \left\| \sum_\alpha \sum_\beta (|\lambda|^n(2|\beta|+n) - \mu)(u, e_{\alpha,\beta}) e^\lambda_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^{2n})} \\
&= \sum_\alpha \sum_\beta ||\lambda|^n(2|\beta|+n) - \mu|^2 \, |(u, e^\lambda_{\alpha,\beta})|^2.
\end{aligned}$$

Thus,

$$\|(L_0^\lambda - \mu I)u\|_{L^2(\mathbb{R}^{2n})} \ge C_\mu \|u\|_{L^2(\mathbb{R}^{2n})}, \quad u \in \mathcal{D}(L_0^\lambda).$$

□

By Theorem X.1 on page 136 of [11] and the preceding theorem, we see that for all $\lambda \in \mathbb{R}^{m*}$, L_0^λ is self-adjoint and hence the λ-twisted Laplacian L_0^λ given by (1.1) from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with dense domain $\mathcal{S}$ is essentially self-adjoint.

12 Sobolev Spaces

Let $s \in \mathbb{R}$. Then for all $\lambda \in \mathbb{R}^{m*}$, we define the L^2-Sobolev space $H^{s,2,\lambda}$ of order s by

$$H^{s,2,\lambda} = \left\{ u \in \mathcal{S}'(\mathbb{R}^{2n}) : \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s} |(u, e^\lambda_{\alpha,\beta})|^2 < \infty \right\}.$$

It is easy to see that $H^{s,2,\lambda}$ is an inner product space with inner product $(\ ,\ )_{s,2,\lambda}$ and norm $\|\ \|_{s,2,\lambda}$ given by

$$(u, v)_{s,2,\lambda} = \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s}(u, e^\lambda_{\alpha,\beta})(e^\lambda_{\alpha,\beta}, v)$$

and

$$\|u\|^2_{s,2,\lambda} = \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s}|(u, e^\lambda_{\alpha,\beta})|^2$$

for all u and v in $H^{s,2,\lambda}$.

Theorem 12.1 *$H^{s,2,\lambda}$ is a Hilbert space with respect to the inner product $(\ ,\ )_{s,2,\lambda}$.*

Proof If $s \geq 0$, then the domain $\mathcal{D}((L_0^\lambda)^s)$ of the self-adjoint operator $(L_0^\lambda)^s$ from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ is a Banach space with respect to the norm $|\ |_s$ given by

$$|u|_s^2 = \|(L_0^\lambda)^s u\|^2_{L^2(\mathbb{R}^{2n})} + \|u\|^2_{L^2(\mathbb{R}^{2n})}, \quad u \in \mathcal{D}((L_0^\lambda)^s).$$

Obviously,

$$\|(L_0^\lambda)^s u\|^2_{L^2(\mathbb{R}^{2n})} = \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s}|(u, e^\lambda_{\alpha,\beta})|^2 = \|u\|^2_{s,2,\lambda}.$$

So, $\| \|_{s,2,\lambda}$ is a norm in $H^{s,2,\lambda}$ and hence $H^{s,2,\lambda}$ is complete with respect to $\| \|_{s,2,\lambda}$. Let $s < 0$. Then $H^{s,2,\lambda}$ is the dual space of $H^{-s,2,\lambda}$ and is hence complete. □

From the proof of the preceding theorem, we can have a characterization of the domain $\mathcal{D}(L_0^\lambda)$ of the closure of the λ-twisted Laplacian.

Theorem 12.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then $\mathcal{D}(L_0^\lambda) = H^{1,2,\lambda}$.*

The following result can be considered to be the analog for the λ-twisted Laplacian of the Agmon–Douglis–Nirenberg inequalities for elliptic boundary-value problems in [1] and globally elliptic pseudo-differential operators on $\mathbb{R}^n$ in [16].

Theorem 12.3 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all $s \in \mathbb{R}$,*

$$\|u\|_{s+1,2,\lambda} = \|L_0^\lambda u\|_{s,2,\lambda}, \quad u \in H^{s+1,2,\lambda}.$$

Proof Let $u \in H^{s+1,2,\lambda}$. Then

$$\begin{aligned}
\|L_0^\lambda u\|_{s,2,\lambda}^2 &= \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta|+n)^{2s}|(L_0^\lambda u, e_{\alpha,\beta}^\lambda)|^2 \\
&= \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta|+n)^{2s}|\lambda|^{2n}(2|\beta|+n)^2)^2|(u, e_{\alpha,\beta}^\lambda)|^2 \\
&= \sum_\alpha \sum_\beta |\lambda|^{2n(s+1)}(2|\beta|+n)^{2(s+1)}|(u, e_{\alpha,\beta})|^2 \\
&= \|u\|_{s+1,2}^2.
\end{aligned}$$

□

We give as a corollary a result on the global regularity of the λ-twisted Laplacian on Sobolev spaces.

Theorem 12.4 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all $s \in \mathbb{R}$,*

$$u \in \mathcal{S}', L^\lambda u \in H^{s,2,\lambda} \Rightarrow u \in H^{s+1,2,\lambda}.$$

Remark 12.5 There is a loss of one derivative globally on $\mathbb{R}^{2n}$ because the operator L_0^λ with $\lambda \in \mathbb{R}^{m*}$ is not globally elliptic on $\mathbb{R}^{2n}$ as defined in [16], notwithstanding its ellipticity at every point in $\mathbb{R}^{2n}$.

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