Initial Value Problems for Heat Equations Generated by Strongly Elliptic (ρ, Λ)-Pseudo-Differential Operators on $\mathbb{R}^n$

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Abstract Using Gårding's inequality for strongly (ρ, Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$, it is shown that these pseudo-differential operators generate strongly continuous one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$. Applications to solutions and mild solutions of initial value problems for heat equations governed by strongly (ρ, Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$ are given.

Key Words pseudo-differential operators, symbols in $S^m_{\rho,\Lambda}$, L^p-boundedness, (ρ, Λ)-ellipticity, strong (ρ, Λ)-ellipticity, Gårding's inequality, Poisson's equations, weak solutions, strong solutions, strongly continuous one-parameter semigroups, Hille–Yosida–Phillips theorem, initial value problems, heat equations, solutions, mild solutions

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1 Introduction

Our starting point is based on the papers [6] and [10]. Let $\Lambda \in C^\infty(\mathbb{R}^n)$ be a positive function such that there exist positive constants for which

$$C_0(1 + |\xi|)^{\mu_0} \le \Lambda(\xi) \le C_1(1 + |\xi|)^{\mu_1}, \quad \xi \in \mathbb{R}^n. \tag{1.1}$$

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Furthermore, we assume that there exists a positive constant μ with $\mu \geq \mu_1$ such that for all multi-indices α, we can find a positive constant C_α, depending on α only, for which

$$|(\partial^\alpha \Lambda)(\xi)| \leq C_\alpha \Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n. \tag{1.2}$$

Let $m \in \mathbb{R}$ and let $\rho \in \left(0, \frac{1}{\mu}\right]$. Then we define $S^m_{\rho,\Lambda}$ to be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{m-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n. \tag{1.3}$$

A function in $S^m_{\rho,\Lambda}$ is said to be a symbol of order m and type ρ with weight Λ. It should be noted that if we let Λ be the weight defined by

$$\Lambda(\xi) = \sqrt{1 + |\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

then $S^m_{\rho,\Lambda}$ is the same as a special case of the Hörmander class $S^m_{\rho,0}$ in [7]. Let $\sigma \in S^m_{\rho,\Lambda}$. Then we define the pseudo-differential operator T_σ associated to the symbol σ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

for all φ in the Schwartz space $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

It can be shown easily that $T_\sigma : \mathcal{S} \to \mathcal{S}$ is a continuous linear mapping.

The following results can be found in the book [8] by Kumano-go. They are analogs of the corresponding results for pseudo-differential operators with symbols in $S^m_{1,0}$ given in the book [11] by Wong.

Theorem 1.1 *Suppose that* $\sigma_j \in S^{m_j}_{\rho,\Lambda}$, *where*

$$m_0 > m_1 > m_2 > \cdots > m_j \to -\infty$$

as $j \to \infty$. *Then there exists a symbol* σ *in* $S^{m_0}_{\rho,\Lambda}$ *such that*

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_j,$$

i.e.,

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in S^{m_N}_{\rho,\Lambda}$$

for every positive integer N. Moreover, if τ is another symbol with the same asymptotic expansion, then $\sigma - \tau \in \bigcap_{m\in\mathbb{R}} S^m_{\rho,\Lambda}$.

Theorem 1.2 *Let $\sigma \in S^{m_1}_{\rho,\Lambda}$ and $\tau \in S^{m_2}_{\rho,\Lambda}$. Then $T_\sigma T_\tau = T_\lambda$, where $\lambda \in S^{m_1+m_2}_{\rho,\Lambda}$ and*

$$\lambda \sim \sum_{\mu} \frac{(-i)^\mu}{\mu!}(\partial^\mu_\xi \sigma)(\partial^\mu_x \tau).$$

Here, the asymptotic expansion means that

$$\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!}(\partial^\mu_\xi \sigma)(\partial^\mu_x \tau) \in S^{m_1+m_2-\rho N}_{\rho,\Lambda}$$

for every positive integer N.

Theorem 1.3 *Let $\sigma \in S^m_{\rho,\Lambda}$. Then the formal adjoint T^*_σ of T_σ is the pseudo-differential operator T_τ, where $\tau \in S^m_{\rho,\Lambda}$ and*

$$\tau \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!}\partial^\mu_x \partial^\mu_\xi \overline{\sigma}.$$

Here, the asymptotic expansion means that

$$\tau - \sum_{|\mu|<N} \frac{(-i)^\mu}{\mu!}\partial^\mu_x \partial^\mu_\xi \overline{\sigma} \in S^{m-\rho N}_{\rho,\Lambda}$$

for every positive integer N.

Using the formal adjoint, we can extend the definition of a pseudo-differential operator from the Schwartz space $\mathcal{S}$ to the space $\mathcal{S}'$ of all tempered distributions. To wit, let $\sigma \in S^m_{\rho,\Lambda}$. Then for all u in $\mathcal{S}'$, we define the linear functional $T_\sigma u$ on $\mathcal{S}$ by

$$(T_\sigma u)(\varphi) = u(\overline{T^*_\sigma \overline{\varphi}}), \quad \varphi \in \mathcal{S}.$$

It is easy to see that T_σ maps $\mathcal{S}'$ into $\mathcal{S}'$ continuously. In fact, we have the following theorem.

Theorem 1.4 *Let* $\sigma \in S^0_{\rho,\Lambda}$. *Then* $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a bounded linear operator.*

The proof of Theorem 1.4 can be found in the book [8] by Kumano-go.

Remark 1.5 If

$$\Lambda(\xi) = \sqrt{1 + |\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

and

$$\rho \in (0, 1),$$

then $S^0_{\rho,\Lambda}$ is the Hörmander class $S^0_{\rho,0}$ and it is well known that Theorem 1.4 cannot be extended to $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Nevertheless, there exists an important subclass of $S^0_{\rho,\Lambda}$, denoted by $M^0_{\rho,\Lambda}$, for which Theorem (1.4) is true for $L^p(\mathbb{R}^n)$ for all p with $1 < p < \infty$. See [2, 3, 6, 10], among others, in this connection.

A symbol σ in $S^m_{\rho,\Lambda}$ is said to be (ρ, Λ)-elliptic if there exist positive constants C and R such that

$$|\sigma(x, \xi)| \geq C\Lambda(\xi)^m, \quad |\xi| \geq R.$$

Theorem 1.6 *Let* $\sigma \in S^m_{\rho,\Lambda}$ *be* (ρ, Λ)*-elliptic. Then there exists a symbol* τ *in* $S^{-m}_{\rho,\Lambda}$ *such that*

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R *and* S *are pseudo-differential operators with symbols in* $\bigcap_{k\in\mathbb{R}} S^k_{\rho,\Lambda}$.

The pseudo-differential operator T_τ in the preceding theorem is known as a parametrix of the (ρ, Λ)-elliptic pseudo-differential operator T_σ.

To extend Theorem 1.4 to pseudo-differential operators with symbols in $S^m_{\rho,\Lambda}$, where m is an arbitrary real number, we need (ρ, Λ)-Sobolev spaces. To this end, we need the following proposition.

Proposition 1.7 $\Lambda \in S^1_{\rho,\Lambda}$.

Proof Let α be a multi-index. Then by (1.2), there exists a positive constant C_α such that

$$|(\partial^\alpha \Lambda)(\xi)| \le C_\alpha \Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

Since $\rho \in \left(0, \frac{1}{\mu}\right]$, it follows that

$$|(\partial^\alpha \Lambda)(\xi)| \le C_\alpha \Lambda(\xi)^{1-\rho|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

Therefore

$$\Lambda \in S^1_{\rho,\Lambda}.$$

$\square$

Let $s \in \mathbb{R}$. Then we define $\Lambda(D)^s$ to be the Fourier multiplier given by

$$\Lambda(D)^s u = \mathcal{F}^{-1}\Lambda^s \mathcal{F} u, \quad u \in \mathcal{S}',$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier transform and the inverse Fourier transform.

For $s \in \mathbb{R}$, we define the Sobolev space $H^{s,2}_\Lambda$ by

$$H^{s,2}_\Lambda = \{u \in \mathcal{S}' : \Lambda(D)^s u \in L^2(\mathbb{R}^n)\}.$$

Then $H^{s,2}_\Lambda$ is a Hilbert space in which the norm $\| \; \|_{s,2,\Lambda}$ is given by

$$\|u\|_{s,2,\Lambda} = \|\Lambda(D)^s u\|_{L^2(\mathbb{R}^n)}, \quad u \in H^{s,2}_\Lambda.$$

The following result is known as the Sobolev embedding theorem.

Theorem 1.8 *For $s \le t$, $H^{t,2}_\Lambda \subseteq H^{s,2}_\Lambda$, and there exists a positive constant C such that*

$$\|u\|_{s,2,\Lambda} \le C\|u\|_{t,2,\Lambda}, \quad u \in H^{t,2}_\Lambda.$$

We have the following extension of Theorem 1.4.

Theorem 1.9 *Let $\sigma \in S^m_{\rho,\Lambda}$. Then for $-\infty < s < \infty$, $T_\sigma : H^{s,2}_\Lambda \to H^{s-m,2}_\Lambda$ is a bounded linear operator.*

Let $\sigma \in S^m_{\rho,\Lambda}$, $m > 0$. Then $T_\sigma : \mathcal{S} \to \mathcal{S}$. So, we can consider T_σ as a linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with dense domain $\mathcal{S}$.

Proposition 1.10 *The linear operator T_σ from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with domain $\mathcal{S}$ is closable.*

A consequence of Proposition 1.10 is that the minimal operator $T_{\sigma,0}$ of T_σ exists. Let us recall that the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions f in $L^2(\mathbb{R}^n)$ for which a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ can be found such that $\varphi_k \to u$ in $L^2(\mathbb{R}^n)$ and $T_\sigma\varphi_k \to f$ in $L^2(\mathbb{R}^n)$ for some f in $L^2(\mathbb{R}^n)$ as $k \to \infty$. Moreover, $T_{\sigma,0}u = f$.

Let u and f be functions in $L^2(\mathbb{R}^n)$. We say that u lies in $\mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = f$ if and only if

$$(u, T_\sigma^*\varphi) = (f, \varphi), \quad \varphi \in \mathcal{S},$$

where

$$(u, v) = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\, dx$$

for all measurable functions u and v on $\mathbb{R}^n$, provided that the integral exists.

Proposition 1.11 *$T_{\sigma,1}$ is a closed linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(T_{\sigma,1})$ containing $\mathcal{S}$.*

Proposition 1.12 *$\mathcal{S} \subseteq \mathcal{D}(T_{\sigma,1})$, where $T_{\sigma,1}^t$ is the true adjoint of $T_{\sigma,1}$.*

Proposition 1.13 *$T_{\sigma,1}$ is an extension of $T_{\sigma,0}$.*

A consequence of Proposition 1.13 is that $T_{\sigma,0}^t$ is an extension of $T_{\sigma,1}^t$. Since $\mathcal{S} \subseteq \mathcal{D}(T_{\sigma,1}^t)$, it follows that $\mathcal{S} \subseteq \mathcal{D}(T_{\sigma,0}^t)$ as well. In this perspective, we have the following result.

Proposition 1.14 *$T_{\sigma,1}$ is the largest closed extension of T_σ having $\mathcal{S}$ contained in the domain of its adjoint. In other words, if B is a closed extension of T_σ such that $\mathcal{S} \subseteq \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B.*

In view of the preceding result, we call $T_{\sigma,1}$ the maximal operator of T_σ. We can prove that $T_{\sigma,0} = T_{\sigma,1}$ if σ is (ρ, Λ)-elliptic. We begin with the following characterization of the domain of $T_{\sigma,0}$.

Theorem 1.15 *If $\sigma \in S^m_{\rho,\Lambda}$ is (ρ, Λ)-elliptic, then $\mathcal{D}(T_{\sigma,0}) = H^{m,2}_\Lambda$.*

To prove Theorem 1.15, we use the following estimate, which is the analog of the Agmon–Douglis–Nirenberg estimate in [1] for pseudo-differential operators. In addition, we use a density result. They are given, respectively, as Theorem 1.16 and Proposition 1.17.

Theorem 1.16 *Let $\sigma \in S^m_{\rho,\Lambda}$ be (ρ,Λ)-elliptic. Then there exist positive constants C_1 and C_2 such that*

$$C_1\|u\|_{m,2,\Lambda} \leq \|T_\sigma u\|_{0,2,\Lambda} + \|u\|_{0,2,\Lambda} \leq C_2\|u\|_{m,2,\Lambda}, \quad u \in H^{m,2}_\Lambda.$$

Proposition 1.17 *For $-\infty < s < \infty$, $\mathcal{S}$ is dense in $H^{s,2}_\Lambda$.*

We can now state the main result in this section.

Theorem 1.18 *Let $\sigma \in S^m_{\rho,\Lambda}$ be (ρ,Λ)-elliptic. Then $T_{\sigma,0} = T_{\sigma,1}$.*

The aim of this paper is to use Gårding's inequality to prove that strongly (ρ,Λ)-elliptic pseudo-differential operators are infinitesimal generators of strongly continuous one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$. In Section 2, we give Gårding's inequality for strongly (ρ,Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$. In Section 3, we aim at giving the key result on the existence and uniqueness of weak (and hence strong) solutions of Poisson's equations modelled by (ρ,Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$. The key result in Section 3 and the Hille–Yosida–Phillips theorem are used in Section 4 to prove that strongly (ρ,Λ)-elliptic pseudo-differential operators generate strongly continuous one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$. We give in Section 5 solutions of initial value problems for heat equations governed by strongly (ρ,Λ)-elliptic pseudo-differential operators.

Related results can be found in [2, 3, 10].

2 Gårding's Inequality

The main result in this section is Gårding's inequality for strongly (ρ,Λ)-elliptic operators. It is an extension of Gårding's inequality in Chapter 17 of the book [11].

Theorem 2.1 (Gårding's Inequality) *Let $\sigma \in S^{2m}_{\rho,\Lambda}$ be such that there exist positive constants C and R for which*

$$\operatorname{Re} \sigma(x,\xi) \geq C\Lambda(\xi)^{2m}, \quad |\xi| \geq R.$$

Then we can find a positive constant C' and a constant C_s for every real number $s \in \left[\frac{\rho}{2}, \infty\right)$ such that

$$\operatorname{Re}(T_\sigma \varphi, \varphi) \geq C' \|\varphi\|^2_{m,2,\Lambda} - C_s \|\varphi\|^2_{m-\rho s,2,\Lambda}, \quad \varphi \in \mathcal{S}.$$

A symbol satisfying the hypothesis of the theorem is said to be strongly (ρ, Λ)-elliptic. In order to prove the theorem, we need some preliminary results, some of which are of interest in their own right.

Lemma 2.2 *Let $F \in C^\infty(\mathbb{C})$. Then for every σ in $S^0_{\rho,\Lambda}$, $F \circ \sigma \in S^0_{\rho,\Lambda}$.*

Proof We need to prove that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta (F \circ \sigma))(x,\xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n. \tag{2.1}$$

(2.1) is true for all multi-indices α and β with $|\alpha + \beta| = 0$. Indeed, there exists a positive constant C such that

$$|\sigma(x,\xi)| \leq C, \quad x, \xi \in \mathbb{R}^n.$$

Thus, $F \circ \sigma$ is in fact a C^∞ function on a compact subset of $\mathbb{R}^{2n}$. Hence there exists another positive constant C' such that

$$|(F \circ \sigma)(x,\xi)| \leq C', \quad x, \xi \in \mathbb{R}^n.$$

Now, suppose that (2.1) is valid for all F in $C^\infty(\mathbb{C})$, σ in $S^0_{\rho,\Lambda}$ and multi-indices α and β with $|\alpha + \beta| = l$. Let α and β be multi-indices with

$$|\alpha + \beta| = l + 1.$$

We first suppose that

$$\partial_x^\alpha \partial_\xi^\beta = \partial_x^\alpha \partial_\xi^\gamma \partial_{\xi_j}$$

for some multi-index γ and some $j = 1, 2, \ldots, n$. Then, by the chain rule,

$$(\partial_x^\alpha \partial_\xi^\beta (F \circ \sigma))(x,\xi) = (\partial_x^\alpha \partial_\xi^\gamma ((F_1 \circ \sigma)\partial_{\xi_j}\sigma + (F_2 \circ \sigma)\partial_{\xi_j}\sigma))(x,\xi)$$

for all x and ξ in $\mathbb{R}^n$, where F_1 and F_2 are the partial derivatives of F with respect to, respectively, the first and second variables. Now, by Leibniz' formula and the induction hypothesis, there exist positive constants $C_{\lambda,\delta}$ and $C_{\alpha,\lambda,\gamma,\delta,j}$ such that

$$\begin{aligned}
& |\partial_x^\alpha \partial_\xi^\gamma ((F_1 \circ \sigma)\partial_{\xi_j}\sigma)(x,\xi)| \\
\leq & \sum_{\lambda\leq\alpha,\delta\leq\gamma} \binom{\alpha}{\lambda}\binom{\gamma}{\delta} |(\partial_x^\lambda \partial_\xi^\delta (F_1\circ\sigma))(x,\xi)|\,|(\partial_x^{\alpha-\lambda}\partial_\xi^{\gamma-\delta}(\partial_{\xi_j}\sigma))(x,\xi)| \\
\leq & \sum_{\lambda\leq\alpha,\delta\leq\gamma} \binom{\alpha}{\lambda}\binom{\gamma}{\delta} C_{\lambda,\delta}\Lambda(\xi)^{-\rho|\delta|} C_{\alpha,\lambda,\gamma,\delta,j}\Lambda(\xi)^{-\rho|\gamma|+\rho|\delta|-\rho} \\
= & C_{\alpha,\gamma,j}\Lambda(\xi)^{-\rho(|\gamma|+1)}, \quad x,\xi\in\mathbb{R}^n,
\end{aligned}$$

where

$$C_{\alpha,\gamma,j} = \sum_{\lambda\leq\alpha,\delta\leq\gamma} \binom{\alpha}{\lambda}\binom{\gamma}{\delta} C_{\lambda,\delta} C_{\alpha,\lambda,\gamma,\delta,j}.$$

Similarly, there exists a positive constant $C'_{\alpha,\gamma,j}$ such that

$$|(\partial_x^\alpha \partial_\xi^\gamma ((F_2\circ\sigma)\partial_{\xi_j}\sigma)(x,\xi)| \leq C'_{\alpha,\gamma,j}\Lambda(\xi)^{-\rho(|\gamma|+1)}$$

for all x and ξ in $\mathbb{R}^n$. Therefore

$$|(\partial_x^\alpha \partial_\xi^\beta (F\circ\sigma))(x,\xi)| \leq (C_{\alpha,\gamma,j} + C'_{\alpha,\gamma,j})\Lambda(\xi)^{-\rho|\beta|}$$

for all x and ξ in $\mathbb{R}^n$. Now, we suppose that

$$\partial_x^\alpha \partial_\xi^\beta = \partial_x^\gamma \partial_{x_j} \partial_\xi^\beta$$

for some multi-index γ and some $j = 1, 2, \ldots, n$. Then as before, there exists a positive constant $C''_{\alpha,\gamma,j}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta (F\circ\sigma))(x,\xi)| \leq C''_{\alpha,\gamma,j}\Lambda(\xi)^{-\rho|\beta|}$$

for all x and ξ in $\mathbb{R}^n$. Thus, by the principle of mathematical induction, (2.1) follows. □

Lemma 2.3 *Let σ be a strongly elliptic symbol in $S^{2m}_{\rho,\Lambda}$. Then there exist positive constants γ and κ such that*

$$\operatorname{Re}\sigma(x,\xi) \geq \gamma\Lambda(\xi)^{2m} - \kappa\Lambda(\xi)^{2m-1}, \quad x,\xi\in\mathbb{R}^n.$$

Proof By strong ellipticity, there exist positive constants C and R such that

$$\operatorname{Re}\sigma(x,\xi) \geq C\Lambda(\xi)^{2m}, \quad |\xi| \geq R.$$

Since $\sigma \in S^{2m}_{\rho,\Lambda}$, we can find a positive constant K such that

$$|\sigma(x,\xi)| \leq K\Lambda(\xi)^{2m}, \quad x,\xi \in \mathbb{R}^n.$$

Therefore

$$|\operatorname{Re}\sigma(x,\xi)| \leq K\Lambda(\xi)^{2m} \leq K(1+R^2)^{2m}, \quad |\xi| \leq R.$$

Hence there exists a positive constant M such that

$$\operatorname{Re}\sigma(x,\xi) \geq -M, \quad |\xi| \leq R.$$

Since $\frac{\operatorname{Re}\sigma}{\Lambda^{2m-1}}$ is a continuous function on the compact set $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, we can find a positive constant κ such that

$$\frac{\operatorname{Re}\sigma(x,\xi)}{\Lambda(\xi)^{2m-1}} \geq -\kappa, \quad |\xi| > R.$$

Therefore

$$\operatorname{Re}\sigma(x,\xi) + \kappa\Lambda(\xi)^{2m-1} > 0, \quad |\xi| \leq R.$$

Since $\frac{\operatorname{Re}\sigma + \kappa\Lambda^{2m-1}}{\Lambda^{2m}}$ is a positive and continuous function on the compact set $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, there is a positive constant δ such that

$$\frac{\operatorname{Re}\sigma(x,\xi) + \kappa\Lambda(\xi)^{2m-1}}{\Lambda(\xi)^{2m}} \geq \delta, \quad |\xi| \leq R.$$

So, the lemma is proved if we let $\gamma = \min(C,\delta)$. □

Proof of Theorem 2.1 Let $T_\tau = \Lambda(D)^{-m} T_\sigma \Lambda(D)^{-m}$. Then using the asymptotic expansion for the product of two pseudo-differential operators,

$$T_\sigma \Lambda(D)^{-m} = T_{\tau_1},$$

where

$$\tau_1 - \Lambda^{-m}\sigma \in S^{m-\rho}_{\rho,\Lambda}. \tag{2.2}$$

Similarly,

$$T_\tau = \Lambda(D)^{-m} T_{\tau_1}$$

and

$$\tau - \Lambda^{-m}\tau_1 \in S^{-\rho}_{\rho,\Lambda}. \tag{2.3}$$

Multiplying (2.2) by Λ^{-m} and adding the result to (2.3), we get

$$\tau - \Lambda^{-2m}\sigma \in S^{-\rho}_{\rho,\Lambda}.$$

Therefore

$$\tau = \Lambda^{-2m}\sigma + r,$$

where $r \in S^{-\rho}_{\rho,\Lambda}$. So, by Lemma 2.3,

$$\begin{aligned}
\operatorname{Re}\tau &= \Lambda^{-2m}\operatorname{Re}\sigma + \operatorname{Re} r \\
&\geq \Lambda^{-2m}[\gamma\Lambda^{2m} - \kappa\Lambda^{2m-\rho}] + \operatorname{Re} r \\
&= \gamma - \kappa\Lambda^{-\rho} + \mathrm{r} \\
&\geq \gamma - \kappa'\Lambda^{-\rho},
\end{aligned}$$

where κ' is another positive constant. Therefore τ satisfies the conclusion of Lemma 2.3 with $m = 0$. Let us suppose for a moment that Gårding's inequality is valid for $m = 0$. Then we can find a positive constant C' and a positive constant C_s for every real number $s \in \left[\frac{\rho}{2}, \infty\right)$ such that

$$\begin{aligned}
\operatorname{Re}(T_\sigma\varphi, \varphi) &= \operatorname{Re}(\Lambda(D)^m T_\tau \Lambda(D)^m\varphi, \varphi) \\
&= \operatorname{Re}(T_\tau\Lambda(D)^m\varphi, \Lambda(D)^m\varphi) \\
&\geq C'\|\Lambda(D)^m\varphi\|^2_{0,2} - C_s\|\Lambda(D)\varphi\|^2_{-\rho s,2} \\
&= C'\|\varphi\|^2_{m,2,\Lambda} - C_s\|\varphi\|^2_{m-\rho s,2,\Lambda}
\end{aligned}$$

for all φ in $\mathcal{S}$. We are now ready to prove Gårding's inequality for $m = 0$. By Lemma 2.3, we have positive constants γ and κ such that

$$\operatorname{Re}\sigma + \kappa\Lambda^{-\rho} \geq \gamma.$$

Let $F \in C^\infty(\mathbb{C})$ be such that

$$F(z) = \sqrt{\frac{\gamma}{2} + z}, \quad z \in [0, \infty).$$

Let τ be the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\tau(x,\xi) = F(2(\operatorname{Re}\sigma(x,\xi) + \kappa\Lambda(\xi)^{-\rho} - \gamma)), \quad x, \xi \in \mathbb{R}^n.$$

Then by Lemma 2.3, $\tau \in S^0_{\rho,\Lambda}$ and for all x and ξ in $\mathbb{R}^n$,

$$\begin{aligned} \tau(x,\xi) &= \sqrt{\frac{\gamma}{2} + 2\operatorname{Re}\sigma(x,\xi) + 2\kappa\Lambda(\xi)^{-\rho} - 2\gamma} \\ &= \sqrt{2\operatorname{Re}\sigma(x,\xi) + 2\kappa\Lambda(\xi)^{-\rho} - \frac{3}{2}\gamma}. \end{aligned}$$

Using the asymptotic expansion for the formal adjoint of a pseudo-differential operator, we have

$$T_\tau^* = T_{\tau^*},$$

where $\tau^* \in S^0_{\rho,\Lambda}$ and $\tau - \tau^* \in S^{-\rho}_{\rho,\Lambda}$. Using also the asymptotic expansion for the product,

$$T_\tau^* T_\tau = T_\lambda,$$

where

$$\lambda - \tau^*\tau \in S^{-\rho}_{\rho,\Lambda}.$$

If we let r_1 and r_1' in $S^{-\rho}_{\rho,\Lambda}$ be such that

$$\tau^* = \tau + r_1$$

and

$$\lambda = \tau^*\tau + r_1',$$

then with $r_2 = r_1\tau + r' \in S^{-\rho}_{\rho,\Lambda}$, we get

$$\begin{aligned} \lambda &= (\tau + r_1)\tau + r_1' \\ &= \tau^2 + r_1\tau + r_1' \\ &= 2\operatorname{Re}\sigma + 2\kappa\Lambda(\xi)^{-\rho} - \frac{3}{2}\gamma + r_2. \end{aligned}$$

So, if we let $r_3 = 2\kappa\Lambda^{-1} + r_2 \in S^{-1}_{\rho,\Lambda}$, then we get

$$\lambda = 2\operatorname{Re}\sigma - \frac{3}{2}\gamma + r_3.$$

But

$$2\mathrm{Re}\,\sigma = \sigma + \overline{\sigma} = \sigma + \sigma^* + r_4$$

for some r_4 in $S^{-\rho}_{\rho,\Lambda}$. Therefore

$$\lambda = \sigma + \sigma^* - \frac{3}{2}\lambda + r_5$$

for some r_5 in $S^{-\rho}_{\rho,\Lambda}$. Thus,

$$\sigma + \sigma^* = \lambda + \frac{3}{2}\gamma + r_5.$$

Now, for all φ in $\mathcal{S}$,

$$\begin{aligned} 2\mathrm{Re}\,(T_\sigma\varphi,\varphi) &= (T_\sigma\varphi,\varphi) + (T^*_\sigma\varphi,\varphi) = (T_{\sigma+\sigma^*}\varphi,\varphi) \\ &= (T_\lambda\varphi,\varphi) + \frac{3}{2}\gamma\|\varphi\|^2_{0,2,\Lambda} + (T_{r_5}\varphi,\varphi) \\ &\geq \gamma\|\varphi\|^2_{0,2,\Lambda} + \left\{\frac{\gamma}{2}\|\varphi\|^2_{0,2,\Lambda} - \|T_{r_5}\varphi\|_{\frac{1}{2},2,\Lambda}\|\varphi\|_{-\frac{1}{2},2,\Lambda}\right\}. \end{aligned}$$

Using the L^2-boundedness of (ρ,Λ)-pseudo-differential operators, we get a positive constant μ such that

$$2\mathrm{Re}\,(T_\sigma\varphi,\varphi) \geq \gamma\|\varphi\|^2_{0,2,\Lambda} + \left\{\frac{\gamma}{2}\|\varphi\|^2_{0,2,\Lambda} - \mu\|\varphi\|^2_{-\frac{1}{2},0,\Lambda}\right\}, \quad \varphi \in \mathcal{S}.$$

But

$$\mu\|\varphi\|^2_{-\frac{1}{2},0,\Lambda} = \int_{\mathbb{R}^n} \mu\Lambda(\xi)^{-1}|\hat{\varphi}(\xi)|^2\,d\xi = I + J,$$

where

$$I = \int_{\mu\Lambda(\xi)^{-1}\leq\frac{\gamma}{2}} \mu\Lambda(\xi)^{-1}|\hat{\varphi}(\xi)|^2 d\xi$$

and

$$J = \int_{\mu\Lambda(\xi)^{-1}\geq\frac{\gamma}{2}} \mu\Lambda(\xi)^{-1}|\hat{\varphi}(\xi)|^2 d\xi.$$

Obviously,

$$I \leq \frac{\gamma}{2}\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 d\xi = \frac{\gamma}{2}\|\varphi\|^2_{0,2}.$$

To estimate J, we note that

$$\mu\Lambda(\xi)^{-1} \geq \frac{\gamma}{2} \Rightarrow \Lambda(\xi) \leq \frac{2\mu}{\gamma}.$$

So, for $\mu\Lambda(\xi)^{-1} \geq \frac{\gamma}{2}$, we get for every nonnegative real number s with $s \geq \frac{\rho}{2}$.

$$\begin{aligned} \mu\Lambda(\xi)^{-1} &= \mu\Lambda(\xi)^{2\rho s-1}\Lambda(\xi)^{-2\rho s} \\ &\leq \mu\left(\frac{2\mu}{\gamma}\right)^{2\rho s-1}\Lambda(\xi)^{-2\rho s}. \end{aligned}$$

So, for every nonnegative real number s with $s \geq \frac{\rho}{2}$,

$$J \leq \mu\left(\frac{2\mu}{\gamma}\right)^{2\rho s-1}\int_{\mathbb{R}^n}\Lambda(\xi)^{-2\rho s}|\hat{\varphi}(\xi)|^2 d\xi = C_s'\|\varphi\|^2_{-\rho s,2,\Lambda},$$

where $C_s' = \mu\left(\frac{2\mu}{\gamma}\right)^{2\rho s-1}$. Therefore

$$2\mathrm{Re}\,(T_\sigma\varphi,\varphi) \geq \gamma\|\varphi\|^2_{0,2,\Lambda} - C_s'\|\varphi\|^2_{-\rho s,2,\Lambda}, \quad \varphi \in \mathcal{S},$$

and the theorem follows with $C_0 = \frac{\gamma}{2}$ and $C_s = \frac{C_s'}{2}$.

3 Poisson's Equations

Let $\sigma \in S^m_{\rho,\Lambda}$, $m > 0$. Let $f \in L^2(\mathbb{R}^n)$. Then a function $u \in L^2(\mathbb{R}^n)$ is said to be a weak solution of the Poisson equation

$$T_\sigma u = f \tag{3.1}$$

on $\mathbb{R}^n$ if $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = f$. A function $u \in L^2(\mathbb{R}^n)$ is said to be a strong solution of the Poisson equation (3.1) if $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. It is obvious that every weak solution is a strong solution. By Theorem 1.18, weak solutions and strong solutions are the same if T_σ is a (ρ,Λ)-elliptic pseudo-differential operator.

The following theorem is a well-known result that follows from the corresponding result in functional analysis. See, for example, Theorem 16.3, in [11].

Theorem 3.1 *Let $\sigma \in S^m_{\rho,\Lambda}$, $m > 0$. Let $f \in L^2(\mathbb{R}^n)$. Then the Poisson equation* (3.1) *has a weak solution u in $L^2(\mathbb{R}^n)$ if and only if there exists a positive constant C such that*

$$|(f,\varphi)| \le C\|T^*_\sigma\varphi\|_2, \quad \varphi \in \mathcal{S},$$

where

$$(f,\varphi) = \int_{\mathbb{R}^n} f(x)\overline{\varphi(x)}\,dx.$$

The focus from now on is on strong solutions in $L^2(\mathbb{R}^n)$ of Poisson's equations modelled by (ρ,Λ)-elliptic pseudo-differential operators.

Theorem 3.2 *Let $\sigma \in S^{2m}_{\rho,\Lambda}$, $m > 0$, be a (ρ,Λ)-elliptic symbol such that there exists a positive constant C for which*

$$\mathrm{Re}\,(T_\sigma\varphi,\varphi) \ge C\|\varphi\|^2_{m,2,\Lambda}, \quad \varphi \in \mathcal{S}. \tag{3.2}$$

Then for every function f in $L^2(\mathbb{R}^n)$, the Poisson equation

$$T_\sigma u = f$$

has a unique strong solution u in $L^2(\mathbb{R}^n)$.

Proof Using the Sobolev embedding theorem, (3.2), and the Cauchy–Schwarz inequality, there exists a positive constant C' such that for all functions φ in $\mathcal{S}$,

$$\|\varphi\|_2^2 \le C'\|\varphi\|^2_{m,2,\Lambda} \le \frac{C'}{C}\|\varphi\|_2\|T^*\varphi\|_2$$

and hence

$$\|\varphi\|_2 \le \frac{C'}{C}\|T^*\varphi\|_2.$$

So, for all functions f in $L^2(\mathbb{R}^n)$,

$$|(f,\varphi)| \le \|f\|_2\|\varphi\|_2 \le \frac{C'}{C}\|f\|_2\|T^*\varphi\|_2, \quad \varphi \in \mathcal{S}.$$

Then by Theorem 3.1, the Poisson equation

$$T_\sigma u = f$$

on $\mathbb{R}^n$ has a weak solution u in $L^2(\mathbb{R}^n)$. By (ρ, Λ)-ellipticity, u is a strong solution in $L^2(\mathbb{R}^n)$. To prove uniqueness, we first note that by (3.2) and a density argument,

$$\mathrm{Re}\,(T_\sigma u, u) \geq C\|u\|_{m,2,\Lambda}^2, \quad u \in H_\Lambda^{2m,2}.$$

Let v be another strong solution in $L^2(\mathbb{R}^n)$ of the Poisson equation

$$T_\sigma u = f$$

on $\mathbb{R}^n$. Then

$$\|u - v\|_{m,2,\Lambda}^2 \leq \frac{1}{C}\mathrm{Re}\,(T_\sigma(u - v), u - v) = 0.$$

Therefore $u = v$ and this completes the proof of the theorem. □

4 Strongly Continuous One-Parameter Semigroups

Let us first prove the following theorem.

Theorem 4.1 *Let $\sigma \in S_{\rho,\Lambda}^{2m,2}$, $m > 0$, be an elliptic symbol such that we can find a positive constant C and a constant λ_0 for which*

$$\mathrm{Re}\,(-T_\sigma\varphi, \varphi) \geq C\|\varphi\|_{m,2,\Lambda}^2 - \lambda_0\|\varphi\|_2^2, \quad \varphi \in \mathcal{S}. \tag{4.1}$$

Let $\lambda > \lambda_0$. Then for every $f \in L^2(\mathbb{R}^n)$, there exists a unique solution $u \in H_\Lambda^{2m,2}$ of the equation

$$(\lambda I - T_{\sigma,0})u = f,$$

where I is the identity operator on $L^2(\mathbb{R}^n)$. Moreover,

$$\|(\lambda I - T_{\sigma,0})u\|_2 \geq (\lambda - \lambda_0)\|u\|_2, \quad u \in H_{\rho,\lambda}^{2m,2}. \tag{4.2}$$

Proof By (4.1), we get

$$\begin{aligned}
\mathrm{Re}\,((\lambda I - T_\sigma)\varphi, \varphi) &= \mathrm{Re}\,((\lambda_0 I - T_\sigma)\varphi, \varphi) + (\lambda - \lambda_0)\|\varphi\|_2^2 \\
&\geq C\|\varphi\|_{m,2,\Lambda}^2 + (\lambda - \lambda_0)\|\varphi\|_2^2 \\
&\geq C\|\varphi\|_{m,2,\Lambda}^2 \qquad (4.3)
\end{aligned}$$

and

$$\operatorname{Re}((\lambda I - T_\sigma)\varphi, \varphi) \geq (\lambda - \lambda_0)\|\varphi\|_2^2 \tag{4.4}$$

for all functions φ in $\mathcal{S}$. By (4.3), Theorem 1.15, and Theorem 3.2, we conclude that the equation

$$(\lambda I - T_{\sigma,0})u = f$$

on $\mathbb{R}^n$ has a unique solution u in $H_\Lambda^{2m,2}$. Moreover, by (4.4) and a limiting argument,

$$\|(\lambda I - T_{\sigma,0})u\|_2 \geq (\lambda - \lambda_0)\|u\|_2, \quad u \in H_\Lambda^{2m,2}.$$

□

We conclude with the following theorem.

Theorem 4.2 *Let $\sigma \in S_{\rho,\Lambda}^{2m}$, $m > 0$, be a strongly (ρ, Λ)-elliptic symbol. Then $-T_{\sigma,0}$ is the infinitesimal generator of a strongly continuous one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$.*

Proof By Gårding's inequality, we can find a positive constant C and a constant C_s for every real number s in $\left[\frac{\rho}{2}, \infty\right)$ such that

$$\operatorname{Re}(T_\sigma\varphi, \varphi) \geq C\|\varphi\|_{m,2,\Lambda}^2 - C_s\|\varphi\|_{m-\rho s,\Lambda}^2, \quad \varphi \in \mathcal{S}.$$

By choosing s such that $\rho s = m$, we get a constant λ_0 such that (4.1) is satisfied. By Theorem 4.1, the resolvent set of the pseudo-differential operator $-T_{\sigma,0}$ is the same as (λ_0, ∞) and

$$\|(-T_{\sigma,0} - \lambda I)^{-1}\|_* \leq (\lambda - \lambda_0)^{-1}, \quad \lambda > \lambda_0,$$

where $\| \|_*$ denotes the norm in the C^*-algebra of bounded linear operators on $L^2(\mathbb{R}^n)$. Hence

$$\|(-T_{\sigma,0} - \lambda I)^{-n}\|_* \leq (\lambda - \lambda_0)^{-n}, \quad n = 1, 2, \ldots.$$

The Hille–Yosida–Phillips theorem then completes the proof. □

The Hille–Yosida–Phillips theorem is one of the basic theorems in one-parameter semigroups of bounded linear operators on Banach spaces. We state it precisely in the followong theorem.

Theorem 4.3 *Let A be a closed linear operator from a complex Banach space X into X with dense domain $\mathcal{D}(A)$. Then A is the infinitesimal generator of a strongly continuous one-parameter semigroup of bounded linear operators on X if and only if we can find a positive number M and a real number ω such that*

$$\{\lambda \in \mathbb{R} : \lambda > \omega\} \subseteq \rho(A)$$

and

$$\|(A - \lambda I)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $\lambda \in (\omega, \infty)$ and all positive integers n, where $\rho(A)$ is the resolvent set of A, I is the identity operator on X and $\|(A - \lambda I)^{-n}\|$ is the operator norm of the n^{th} power of $(A - \lambda I)^{-1}$.

See, for instance, [4, 5, 9, 11] in this connection.

5 Initial Value Problems for Heat Equations

Let us now consider the following initial value problem for the heat equation governed by a strongly (ρ, Λ)-elliptic pseudo-differential operator, *i.e.,*

$$\begin{cases} \frac{\partial u}{\partial t} = -T_{\sigma,0}(u(t)), \quad t > 0, \\ u(0) = f, \end{cases} \tag{5.1}$$

where $\sigma \in S^{2m}_{\rho,\Lambda}$ is a strongly (ρ, Λ)-elliptic symbol with $m > 0$, f is a given function in $L^2(\mathbb{R}^n)$, and $u : [0, \infty) \to L^2(\mathbb{R}^n)$ is to be determined.

A solution u of the initial value problem (5.1) is a continuous function $u : [0, \infty) \to L^2(\mathbb{R}^n)$ such that u is continuously differentiable on $(0, \infty)$ and satisfies both equations in (5.1). Using Theorem 4.2 ensuring the existence of a strongly continuous one-parameter semigroup generated by $-T_{\sigma,0}$ and Theorem 1.3 in Chapter 4 of [9], we have the following theorem.

Theorem 5.1 *The initial value problem* (5.1) *has a unique solution u if and only if $f \in H^{2m,2}_{\Lambda}$. Moreover,*

$$u(t) = e^{-T_{\sigma,0}t} f, \quad t \in [0, \infty).$$

If $f \in L^2(\mathbb{R}^n) \setminus H_\Lambda^{2m,2}$, then the function $u : [0, \infty) \to L^2(\mathbb{R}^n)$ defined by

$$u(t) = e^{-T_{\sigma,0}t} f, \quad t \in [0, \infty), \tag{5.2}$$

is merely a mild solution of the initial value problem (5.1) expressed in the following theorem. Briefly put, the function (5.2) is not a solution of the initial value problem (5.1) in the sense that we want it to be.

Theorem 5.2 *Let $\{T(t) : t > 0\}$ be the strongly continuous one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$ generated by $-T_{\sigma,0}$. Then the function $u : [0, \infty) \to L^2(\mathbb{R}^n)$ defined by*

$$u(t) = T(t)f, \quad t \geq 0,$$

is a mild solution of the initial value problem (5.1) in the sense that there exists a sequence $\{f_k\}_{k=1}^\infty$ in $H_\Lambda^{2m,2}$ such that $f_k \to u(0)$ in $L^2(\mathbb{R}^n)$ and $T(\cdot)f_k \to u$ in $L^2(\mathbb{R}^n)$ uniformly on every compact subset of $[0, \infty)$ as $k \to \infty$.

Proof Since $H_\Lambda^{2m,2}$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k=1}^\infty$ in $H_\Lambda^{2m,2}$ such that $f_k \to f$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. So, $f_k \to u(0)$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. By Theorem 2.2 in Chapter 1 of [9], we can find constants ω and M with $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\|_* \leq Me^{\omega t}, \quad t \in [0, \infty).$$

Let K be a compact subset of $[0, \infty)$. Then for all t in K,

$$\|T(t)f_k - u(t)\|_2 = \|T(t)f_k - T(t)f\|_2 \leq \|T(t)\|_* \|f_k - f\|_2 \leq Me^{\omega t}\|f_k - f\|_2.$$

Let

$$S = \sup_{t \in K} e^{\omega t}.$$

Then

$$\|T(t)f_k - T(t)f\|_2 \leq MS\|f_k - f\|_2 \to 0$$

for all $t \in K$ as $k \to \infty$. Therefore

$$T(\cdot)f_k \to u$$

in $L^2(\mathbb{R}^n)$ uniformly on every compact subset of $[0, \infty)$ as $k \to \infty$. This completes the proof that $u : [0, \infty) \to L^2(\mathbb{R}^n)$ is a mild solution of the initial value problem (5.1). $\square$

Remark 5.3 It is clear that the proof of Theorem 5.2 is independent of the choice of the sequence $\{f_k\}_{k=1}^{\infty}$ in $H_\Lambda^{2m,2}$. So the mild solution is unique.

Confirmation This is the confirmation from the corresponding author that the two listed authors are the only authors and each author has agreed to the content, equal contributions to the research, writing and checking on the content of this paper.

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