HILBERT–SCHIMDT AND TRACE CLASS PSEUDO-DIFFERENTIAL OPERATORS AND WEYL TRANSFORMS ON THE AFFINE GROUP

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Abstract. We give necessary and sufficient conditions on the symbols for which the corresponding pseudo-differential operators on the affine group are Hilbert–Schimdt operators. We also give a characterization of trace class pseudo-differential operators on the affine group. A trace formula for these trace class operators are also obtained. We have also obtained the L^2 boundedness of the Weyl transforms on the affine group.

1. Introduction

The classical pseudo-differential operator T_σ associated to a measurable function σ on $\mathbb{R}^n \times \mathbb{R}^n$ is defined by

$$(T_\sigma\varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\widehat{\varphi}(\xi)d\xi,\ x \in \mathbb{R}^n,$$

for all Schwartz functions φ on $\mathbb{R}^n$, provided that the integral exists. The function $\widehat{\varphi}$ in the above formula is the Fourier transform of the function φ defined by

$$\widehat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\, d\xi,\ \xi \in \mathbb{R}^n.$$

The genesis of pseudo-differential operators defined above is based on the Fourier inversion formula for the Fourier transform and is done by inserting a symbol on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ in the Fourier inversion formula. Here, the second $\mathbb{R}^n$ in the product $\mathbb{R}^n \times \mathbb{R}^n$ is the dual group of $\mathbb{R}^n$. Using this idea, the study of pseudo-differential operators has been extended to other groups where the dual group and the Fourier inversion formula are explicitly known. See, for instance, [2, 3, 5, 6, 11], among others.

For any locally compact and Hausdorff group G, the set of equivalence classes of strongly continuous, irreducible and unitary representations is known as the dual of G and is denoted by $\widehat{G}$. If G is noncompact then the dual may be infinite-dimensional as in the case of $\mathbb{R}^n$ and the Heisenberg group. In general, the Fourier transform of any function in $L^1(G)$ is an operator-valued function on the dual $\widehat{G}$ and the symbol

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of the corresponding pseudo-differential operator is an operator-valued function on $G \times \widehat{G}$. These operators have many applications in quantum physics [7].

The aim of this paper is to extend the analysis of pseudo-differential operators on the affine group studied in [1]. In Section 2 we recall the basics of the affine group and the Fourier analysis on the affine group. We recall the L^2-boundedness result of pseudo-differential operators on the affine group and prove the equality of pseudo-differential operators with equal symbols in Section 3. In Section 4 we characterize the symbols for which these operators are Hilbert–Schimdt operators. In Section 5 we obtain the trace formula for the trace class pseudo-differential operators on the affine group. We also give the Fourier–Wigner tranansforms and the Weyl transforms in Section 6.

2. The Affine Group

Let U be the upper half plane defined by

$$U = \{(b, a) : b \in \mathbb{R}, a > 0\}.$$

Then U is group with the binary operation $\cdot$ defined by

$$(b, a) \cdot (c, d) = (b + ac, ad) \tag{2.1}$$

for all $(b, a), (c, d) \in U$. With respect to the multiplication $\cdot$ given in (2.1), one can show that U is a non-abelian group. It can be shown that $(-\frac{b}{a}, \frac{1}{a})$ is the inverse element of (b, a) and $(0, 1)$ is the identity element in U. The left and right Haar measures on U are given by $d\mu = \frac{db\,da}{a^2}$ and $d\nu = \frac{db\,da}{a}$, respectively.

With respect to the above multiplication $\cdot$ defined by (2.1), U is also a locally compact and Hausdroff group on which the left Haar measure is different from the right Haar measure. Thus, U is a non-unimodular group, which is known as the affine group.

Let $H^2_+(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined by

$$H^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subseteq [0, \infty)\},$$

where $\mathrm{supp}(\widehat{f})$ is the set of all $x \in \mathbb{R}$ for which there is no neighborhood of x on which $\widehat{f}$ is equal to zero almost everywhere. Similarly, $H^2_-(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is defined by

$$H^2_-(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subseteq (-\infty, 0]\}.$$

It can be proved that $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ are closed subspace of $L^2(\mathbb{R})$. The spaces $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ are known as the Hardy space and the conjugate Hardy space, respectively.

Let $U(H^2_\pm(\mathbb{R}))$ be the set of all unitary operators on $H^2_\pm(\mathbb{R})$. It is a group with respect to the composition of mappings. Then the unitary and irreducible representations of U on $H_\pm(\mathbb{R})$ are given by the mapping $\pi_\pm : U \to U(H^2_\pm(\mathbb{R}))$ defined as

$$(\pi_\pm(b,a)f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x-a}{b}\right),\ x \in \mathbb{R},$$

for all points (b,a) in U and all functions $f \in H^2_\pm(\mathbb{R})$. More details on the affine group and its representations can be found in [1, 9, 10], among others.

To describe the Fourier analysis on the affine group, we look at the equivalent representations of $\pi_\pm : U \to U(H^2_\pm(\mathbb{R}))$, denoted by $\rho_\pm : U \to U(L^2(\mathbb{R}_\pm))$, given by

$$(\rho_+(b,a)u)(s) = a^{1/2} e^{-ibs} u(as),\ s \in \mathbb{R}_+ = [0,\infty),$$

for all $u \in L^2(\mathbb{R}_+)$, and

$$(\rho_-(b,a)u)(s) = a^{1/2} e^{-ibs} v(as),\ s \in \mathbb{R}_- = (-\infty, 0],$$

for all $v \in L^2(\mathbb{R}_-)$. We recall the Duflo-Moore operators $D_\pm$ [4], which are unbounded operators on $L^2(\mathbb{R}_\pm)$, defined by

$$(D_\pm \varphi)(s) = |s|^{1/2} \varphi(s),\ s \in \mathbb{R}_\pm.$$

Then for all $f \in L^2(U)$, the Fourier transform $\widehat{f}$ of f is the function on $\{\rho_+, \rho_-\}$ defined by

$$(\widehat{f}(\rho_\pm)\psi)(x) = \int_0^\infty \int_{-\infty}^\infty f(b,a)(\rho_\pm(b,a) D_\pm \psi)(x) \frac{db\, da}{a^2},\ x \in \mathbb{R}_\pm,$$

for all $\psi \in L^2(\mathbb{R}_\pm)$. Then the Plancheral formula states that

$$||\widehat{f}(\rho_+)||^2_{S^2} + ||\widehat{f}(\rho_-)||^2_{S^2} = ||f||^2_{L^2(U)} \tag{2.2}$$

for all $f \in L^2(U)$, where $||\ ||_{S^2}$ is the Hilbert–Schimdt norm. The Fourier inversion formula states that for all $f \in L^2(U)$, we get

$$f(b,a) = \frac{\sqrt{a}}{2\pi} \mathrm{tr}(D_+ \widehat{f}(\rho_+)\rho_+(b,a)^*) + \frac{\sqrt{a}}{2\pi} \mathrm{tr}(D_- \widehat{f}(\rho_-)\rho_-(b,a)^*)$$

for all $(b,a) \in U$.

Denoting $\{\rho_+, \rho_-\}$ by $\{\pm\}$, we consider the mappings $\sigma : U \times \{\pm\} \to B(L^2(\mathbb{R}))$, where $B(L^2(\mathbb{R}))$ is the C^* algebra of all bounded linear operators on $L^2(\mathbb{R})$. Then for all $f \in L^2(U)$, the pseudo-differential operator T_σ on the affine group U is defined by

$$(T_\sigma f)(b,a) = \frac{\sqrt{a}}{2\pi} \sum_{j=\pm} \mathrm{tr}(\sigma(b,a,j) D_j \widehat{f}(\rho_j)\rho_j(b,a)^*),\ (b,a) \in U. \tag{2.3}$$

Now, after a simple calculation, the Fourier transform of any function $f \in L^2(U)$ can be expressed as

$$(\widehat{f}(\rho_+)\psi)(x) = \int_0^\infty K_f^+(x,y)\psi(y)\, dy$$

for all $\psi \in L^2(\mathbb{R}_+)$, where

$$K_f^+(x,y) = \frac{\sqrt{x}}{y} \int_{-\infty}^\infty f\left(b, \frac{y}{x}\right) e^{-ibx} db = \sqrt{2\pi} \frac{\sqrt{x}}{y} (\mathcal{F}_1 f)\left(x, \frac{y}{s}\right),\ 0 < x, y < \infty, \tag{2.4}$$

and

$$(\widehat{f}(\rho_-)\psi)(x) = \int_{-\infty}^0 K_f^-(x,y)\psi(y)\, dy$$

for all $\psi \in L^2(\mathbb{R}_-)$, where

$$K_f^-(x,y) = \sqrt{2\pi}\frac{\sqrt{|x|}}{|y|}(\mathcal{F}_1 f)\left(x, \frac{y}{s}\right), \; -\infty < x, y < 0. \tag{2.5}$$

Now, for $f \in L^2(U)$, the operator $\widehat{f}(\rho) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$\widehat{f}(\rho)\psi = \widehat{f}(\rho_+)\psi_+ + \widehat{f}(\rho_-)\psi_-, \tag{2.6}$$

where

$$\psi_\pm = \psi\chi_{\mathbb{R}_\pm}.$$

Here,

$$\chi_{\mathbb{R}_\pm}(s) = \begin{cases} 1, & s \in \mathbb{R}_\pm, \\ 0, & s \notin \mathbb{R}_\pm. \end{cases}$$

Then we recall the following result from [1].

Theorem 2.1. *Let $f \in L^2(U)$. Then for all $\psi \in L^2(\mathbb{R})$,*

$$\widehat{f}(\rho)\psi = W_{\sigma_f}\psi,$$

where

$$\sigma_f(x,y) = \frac{1}{\sqrt{2\pi}}(\mathcal{F}_2 T K_f)(x,y),$$

$$K_f(x,y) = \begin{cases} K_f^+(x,y), & x > 0, y > 0, \\ K_f^-(x,y), & x < 0, y < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Moreover,

$$\sigma_f^+(x,y) = \frac{1}{\sqrt{2\pi}}(\mathcal{F}_2 T K_f^+)(x,y),$$

$$\sigma_f^-(x,y) = \frac{1}{\sqrt{2\pi}}(\mathcal{F}_2 T K_f^-)(x,y),$$

where T is the twisting operator defined by

$$(Tf)(x,y) = f\left(x + \frac{y}{2}, x - \frac{y}{2}\right), \; x, y \in \mathbb{R}. \tag{2.8}$$

Moreover, it has been shown in [1] that the Fourier transform on the affine group is a Weyl transform on $L^2(\mathbb{R})$.

Theorem 2.2. *Let $f \in L^2(U)$. Then for all $\varphi \in L^2(\mathbb{R})$,*

$$\widehat{f}(\rho)\varphi = W_{\sigma_f}\varphi, \; \varphi \in L^2(\mathbb{R}),$$

where

$$\sigma_f(x,\xi) = (2\pi)^{-1/2}(\mathcal{F}_2 T K_f)(x,\xi), \; x, \xi \in \mathbb{R},$$

where T is the twisting operator defined by (2.8)*.*

3. L^2-Boundedness

In this section we first recall the from [1] the L^2-boundedness of pseudo-differential operators on U and prove that under suitable conditions, two symbols giving the same pseudo-differential operator are equal.

Theorem 3.1. *Let $\sigma : U \times \{\pm\} \to S_2$ be such that*

$$\sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty ||\sigma(b,a,j)D_j||_{S_2}^2 \frac{db\,da}{a} < \infty.$$

Then $T_\sigma : L^2(U) \to L^2(U)$ is a bounded linear operator. Moreover,

$$||T_\sigma||_* \leq \left\{ \sum_{j=\pm} \int_{-\infty}^\infty ||\sigma(b,a,j)D_j||_{S_2}^2 \frac{db\,da}{a} \right\}^{1/2}.$$

Next, we prove the theorem for equality of symbols.

Theorem 3.2. *Let $\sigma : U \times \{\pm\} \to S_2$ be an operator-valued symbol such that*

$$\sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{s_2}^2 \frac{db\,da}{a} < \infty \tag{3.1}$$

and the mapping

$$U \times \{\pm\} \ni (b,a,j) \mapsto \rho_\pm^*(b,a)\sigma(b,a,\pm) \in S_2$$

is weakly continuous. Then $T_\sigma f = 0$ for all $f \in L^2(U)$ only if $\sigma(b,a,\pm) = 0$ for almost all $(b,a,\pm) \in U \times \{\pm\}$.

Proof. We know that for all $f \in L^2(U)$,

$$\begin{aligned} &(T_\sigma f)(b,a) \\ = \; &\frac{\sqrt{a}}{2\pi}[\mathrm{tr}(\sigma(b,a,+)D_+\widehat{f}(\rho_+)\rho_+ \mathrm{tr}(\sigma(b,a,-)D_-\widehat{f}(\rho_-)\rho_-(b,a)^*)] \end{aligned} \tag{3.2}$$

and

$$f(b,a) = \frac{\sqrt{a}}{2\pi}\mathrm{tr}(D_+\widehat{f}(\rho_+)\rho_+(b,a)^*) + \frac{\sqrt{a}}{2\pi}\mathrm{tr}(D_-\widehat{f}(\rho_-)\rho_-(b,a)^*) \tag{3.3}$$

for all (b,a) in U. Let $(b,a) \in U$. Then we define the function $f_{b,a}$ in $L^2(U)$ by

$$\widehat{f_{b,a}}(\rho_+) = (\sigma(b,a,+)D_+)^*\rho_+(b,a) \tag{3.4}$$

and

$$\widehat{f_{b,a}}(\rho_-) = (\sigma(b,a,-)D_-)^*\rho_-(b,a). \tag{3.5}$$

Now, by the Fourier inversion formula,

$$f_{b,a}(c,d) = \frac{\sqrt{d}}{2\pi}\mathrm{tr}(D_+\widehat{f}_{(b,a)}(\rho_+)\rho_+(c,d)^*) + \frac{\sqrt{d}}{2\pi}\mathrm{tr}(D_-\widehat{f}_{(b,a)}(\rho_-)\rho_-(c,d)^*) \tag{3.6}$$

for all (c, d) in U. Then by the definition of pseudo-differential operators, we get

$$(T_\sigma f_{b,a})(c,d) = \frac{\sqrt{d}}{2\pi}\mathrm{tr}(\sigma(c,d,+)D_+\hat{f}_{b,a}(\rho_+)\rho_+(c,d)^*) + \frac{\sqrt{d}}{2\pi}\mathrm{tr}(\sigma(c,d,-)D_-\hat{f}_{b,a}(\rho_-)\rho_-(c,d)^*)$$

for all (c, d) in U. So, for all $(c, d) \in U$,

$$(T_\sigma f_{b,a})(c,d) = \frac{\sqrt{d}}{2\pi}\mathrm{tr}(\sigma(c,d,+)D_+(\sigma(b,a,+)D_+)^*\rho_+(b,a)\rho_+(c,d)^*) + \frac{\sqrt{d}}{2\pi}\mathrm{tr}(\sigma(c,d,-)D_-(\sigma(b,a,-)D_-)^*\rho_-(b,a)\rho_-(c,d)^*).$$

Since the mapping $U \times \{\pm\} \ni (b, a, j) \mapsto \rho_j^*(b, a)\sigma(b, a, \pm)D_\pm \in S_2$ is weakly continuous, it follows that as $(c, d) \to (b, a)$ we have

$$\mathrm{tr}(\sigma(c,d,+)D_+(\sigma(b,a,+)D_+)^*\rho_+(b,a)\rho_+(c,d)^*) + \mathrm{tr}(\sigma(c,d,-)D_-(\sigma(b,a,-)D_-)^*\rho_-(b,a)\rho_-(c,d)^*)$$

$$\longrightarrow$$

$$\mathrm{tr}(\sigma(b,a,+)D_+(\sigma(b,a,+)D_+)^*\rho_+(b,a)\rho_+(b,a)^*) + \mathrm{tr}(\sigma(b,a,-)D_-(\sigma(b,a,-)D_-)^*\rho_-(b,a)\rho_-(b,a)^*)$$

$$=$$

$$\mathrm{tr}(\sigma(b,a,+)D_+(\sigma(b,a,+)D_+)^*) + \mathrm{tr}(\sigma(b,a,-)D_-(\sigma(b,a,-)D_-)^*)$$

and hence

$$\begin{aligned} &(T_\sigma f_{b,a})(b,a) \\ = \;& \frac{\sqrt{a}}{2\pi}[\mathrm{tr}(\sigma(b,a,+)D_+(\sigma(b,a,+)D_+)^*) + \mathrm{tr}(\sigma(b,a,-)D_-(\sigma(b,a,-)D_-)^*)] \\ = \;& \|\sigma(b,a,+)D_+\|_{s_2}^2 + \|\sigma(b,a,-)D_-\|_{s_2}^2 = 0 \end{aligned}$$

for all (b, a) in U. Thus, $\|\sigma(b, a, \pm)D_\pm\|_{s_2}^2 = 0$ and we get $\sigma(b, a, \pm)D_\pm = 0$ for almost all $(b, a) \in U$. Since $D_\pm$ is injective, $\sigma(b, a, \pm) = 0$ for almost all $(b, a) \in U$. □

4. Hilbert–Schimdt Pseudo-Differential Operators

In this section we first recall the twisting operator [8] and then characterize the Hilbert–Schimdt pseudo-differential operators on the affine group U. Let $T : L^2(\mathbb{R} \times \mathbb{R}) \to L^2(\mathbb{R} \times \mathbb{R})$ be defined by

$$(Tf)(x,y) = f\left(x + \frac{y}{2}, x - \frac{y}{2}\right), \; x, y \in \mathbb{R}.$$

Then $T : L^2(\mathbb{R} \times \mathbb{R}) \to L^2(\mathbb{R} \times \mathbb{R})$ is a bounded linear operator and is usually called the twisting operator. To get a formula for the adjoint T^* of T, we note that for all functions f and g in $L^2(U)$,

$$(Tf, g)_{L^2(\mathbb{R}\times\mathbb{R})} = (f, T^*g)_{L^2(\mathbb{R}\times\mathbb{R})}.$$

Now,

$$(f, T^*g)_{L^2(\mathbb{R}\times\mathbb{R})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\overline{g(x,y)}dx\,dy.$$

Putting $(x+\frac{y}{2}, x-\frac{y}{2}) = (\xi, \eta)$, we get

$$(f, T^*g)_{L^2(\mathbb{R}\times\mathbb{R})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\overline{g\left(\frac{\xi+\eta}{2}, \xi-\eta\right)}d\xi\,d\eta, \tag{4.1}$$

which is the same as

$$(f, T^*g)_{L^2(\mathbb{R}\times\mathbb{R})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{g\left(\frac{x+y}{2}, x-y\right)}dx\,dy. \tag{4.2}$$

Hence for all $g \in L^2(\mathbb{R}\times\mathbb{R})$,

$$(T^*g)(x,y) = g\left(\frac{x+y}{2}, x-y\right),\ \ x, y \in \mathbb{R},$$

which gives

$$(TT^*g)(x,y) = (T^*Tg)(x,y) = g(x,y),\ \ x, y \in \mathbb{R}. \tag{4.3}$$

Theorem 4.1. *Let $\sigma : U \times \{\pm\} \to S_2$ be defined by*

$$\sigma(b,a,j)D_j = \rho_j(b,a)W_{\tau_{\alpha(b,a)}}, \quad j = \pm, \tag{4.4}$$

where

$$\tau_{\alpha(b,a)}(x,\xi) = \mathcal{F}_2^{-1}TK_{\alpha(b,a)}(x,\xi) \tag{4.5}$$

and

$$K_{\alpha(b,a)}(x,\xi) = \begin{cases} \frac{\sqrt{x}}{\xi}\mathcal{F}_1^{-1}\alpha(b,a)\left(x, \frac{\xi}{x}\right), & x>0, \xi>0,\\ \frac{\sqrt{|x|}}{|\xi|}\mathcal{F}_1^{-1}\alpha(b,a)\left(x, \frac{\xi}{x}\right), & x<0, \xi<0\ ,\\ 0, & \text{otherwise,}\end{cases} \tag{4.6}$$

and the mapping $\alpha : U \to L^2(U)$ satisfies the condition

$$\frac{1}{4\pi^2}\int_U \|\alpha(b,a)\|_{s_2}^2\frac{dbda}{a} < \infty.$$

Then $T_\sigma : L^2(U) \to L^2(U)$ is a Hilbert–Schmidt operator and vice-versa.

Proof. We know that the pseudo-differential operator T_σ on $C_0^\infty(U)$ is defined by

$$(T_\sigma f)(b,a) = \frac{\sqrt{a}}{2\pi}\sum_{j=\pm}\mathrm{tr}(\sigma(b,a,j)D_j\widehat{f}(\rho_j)\rho_j(b,a)^*)$$

for all $(b, a) \in U$. Now, using the Parseval identity and the fact that T is a unitary operator, we get

$$\begin{aligned}
& \operatorname{tr}(\rho_+(b,a)^*\sigma(b,a,+)D_+\hat{f}(\rho_+)) \\
= & \operatorname{tr}(W_{\tau_{\alpha(b,a)}} W_{\sigma_f^+}) \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_f^+(x,\xi)\tau_{\alpha(b,a)}(x,\xi)dx\,d\xi \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}_2 T K_f^+(x,\xi)\mathcal{F}_2^{-1} T K_{\alpha(b,a)}(x,\xi)dx\,d\xi \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T K_f^+(x,\xi) T K_{\alpha(b,a)}(x,\xi)dx\,d\xi \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_f^+(x,\xi) K_{\alpha(b,a)}(x,\xi)dx\,d\xi \\
= & \int_0^{\infty}\int_0^{\infty} \frac{\sqrt{x}}{\xi}(\mathcal{F}_1 f)\left(x,\frac{\xi}{x}\right)\frac{\sqrt{x}}{\xi}\mathcal{F}_1^{-1}\alpha(b,a)\left(x,\frac{\xi}{x}\right)dx\,d\xi \\
= & \int_0^{\infty}\int_0^{\infty} \frac{x}{\xi^2}(\mathcal{F}_1 f)\left(x,\frac{\xi}{x}\right)\mathcal{F}_1^{-1}\alpha(b,a)\left(x,\frac{\xi}{x}\right)dx\,d\xi \\
= & \int_0^{\infty}\int_0^{\infty} (\mathcal{F}_1 f)(x,t)\mathcal{F}_1^{-1}\alpha(b,a)(x,t)\frac{dx\,dt}{t^2} \qquad (4.7)
\end{aligned}$$

for all $(b, a) \in U$. Similarly,

$$\operatorname{tr}(\rho_-(b,a)^*\sigma(b,a,-)D_-\hat{f}(\rho_-)) = \int_{-\infty}^{0}\int_0^{\infty} (\mathcal{F}_1 f)(x,t)\mathcal{F}_1^{-1}\alpha(b,a)(x,t)\frac{dx\,dt}{t^2} \qquad (4.8)$$

for all $(b, a) \in U$. Adding the two equation (4.7) and (4.8),

$$\begin{aligned}
(T_\sigma f)(b,a) &= \frac{\sqrt{a}}{2\pi}\int_0^{\infty}\int_0^{\infty} (\mathcal{F}_1 f)(x,t)\mathcal{F}_1^{-1}\alpha(b,a)(x,t)\frac{dx\,dt}{t^2} \\
&+ \frac{\sqrt{a}}{2\pi}\int_{-\infty}^{0}\int_0^{\infty} (\mathcal{F}_1 f)(x,t)\mathcal{F}_1^{-1}\alpha(b,a)(x,t)\frac{dx\,dt}{t^2} \\
&= \frac{\sqrt{a}}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} (\mathcal{F}_1 f)(x,t)\mathcal{F}_1^{-1}\alpha(b,a)(x,t)\frac{dx\,dt}{t^2} \\
&= \frac{\sqrt{a}}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(x,t)\alpha(b,a)(x,t)\frac{dx\,dt}{t^2}
\end{aligned}$$

for all $(b, a) \in U$. Then the kernel k of T_σ is the function on $U \times U$ given by

$$k(b,a,x,t) = \frac{\sqrt{a}}{2\pi}\alpha(b,a)(x,t), \ (b,a),(x,t) \in U. \qquad (4.9)$$

Now,

$$\begin{aligned}
&\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^\infty |k(b,a,x,t)|^2\frac{db\,da}{a^2}\frac{dx\,dt}{t^2}\\
=\;&\int_0^\infty\int_{-\infty}^\infty\int_{-0}^\infty\int_{-\infty}^\infty \frac{a}{4\pi^2}|\alpha(b,a)(x,t)|^2\frac{db\,da}{a^2}\frac{dx\,dt}{t^2}\\
=\;&\frac{1}{4\pi^2}\int_0^\infty\int_{-\infty}^\infty \|\alpha(b,a)\|^2_{L^2(U)}\frac{db\,da}{a}<\infty.
\end{aligned}\tag{4.10}$$

Thus, $T_\sigma : L^2(U) \to L^2(U)$ is a Hilbert–Schimdt operator. Conversely, let $T_\sigma : L^2(U) \to L^2(U)$ be a Hilbert–Schimdt operator. Then

$$(T_\sigma f)(b,a)=\int_0^\infty\int_{-\infty}^\infty \alpha(b,a,x,t)f(x,t)\frac{dx\,dt}{t^2},\quad (b,a)\in U,\tag{4.11}$$

for all $f \in L^2(U)$, where $\alpha \in L^2(U \times U)$. Let $\beta : U \to L^2(U)$ be the mapping defined by

$$\beta(b,a)(x,t)=\frac{\sqrt{a}}{2\pi}\alpha(b,a,x,t),\quad (b,a),(x,t)\in U.$$

Reversing the proof of the sufficiency with β instead of α, we get

$$(T_\sigma f)(b,a)=\frac{\sqrt{a}}{2\pi}\mathrm{tr}(W_{\tau_{\beta(b,a)}}W^+_{\sigma_f})+\frac{\sqrt{a}}{2\pi}\mathrm{tr}(W_{\tau_{\beta(b,a)}}W^-_{\sigma_f}),\quad (b,a)\in U,\tag{4.12}$$

with $\tau_{\beta(b,a)} = \mathcal{F}_2^{-1}TK_{\beta(b,a)}$, where T and K are as defined in the statement of the theorem. Using the assumption that T_σ is a Hilbert–Schimdt operator, it is immediate that

$$\int_0^\infty\int_{-\infty}^\infty ||\beta(b,a)||_{S_2}\frac{db\,da}{a^2}<\infty.$$

But any pseudo-differential operator on the affine group is of the form,

$$(T_\sigma f)(b,a)=\frac{\sqrt{a}}{2\pi}\mathrm{tr}\sum_{j=\pm}(\rho_j(b,a)^*\sigma(b,a,j)D_j\hat{f}(\rho_j))\tag{4.13}$$

for all $(b,a) \in U$ Subtracting (4.13) from (4.12), we get

$$\frac{\sqrt{a}}{2\pi}\mathrm{tr}\sum_{j=\pm}(\rho_j(b,a)^*\sigma(b,a,j)D_j\hat{f}(\rho_+)-W_{\tau_{\beta(b,a)}}W_{\sigma_f^j})=0,\quad (b,a)\in U,$$

This gives

$$\frac{\sqrt{a}}{2\pi}\mathrm{tr}\sum_{j=\pm}[(\rho_j(b,a)^*\sigma(b,a,+)D_j-W_{\tau_{\beta(b,a)}})\hat{f}(\rho_j)=0,\quad (b,a)\in U.$$

By Theorem (3.2), we get for all $(b,a) \in U$,

$$\rho_+(b,a)^*\sigma(b,a,+)D_+-W_{\tau_{\beta(b,a)}}=0$$

and

$$\rho_-(b,a)^*\sigma(b,a,-)D_--W_{\tau_{\beta(b,a)}}=0.$$

So, for all $(b,a) \in U$,

$$\rho_+(b,a)^*\sigma(b,a,+)D_+=W_{\tau_{\beta(b,a)}}$$

and

$$\rho_-(b,a)^*\sigma(b,a,-)D_- = W_{\tau_{\beta(b,a)}}.$$

Hence

$$\sigma(b,a,\pm)D_\pm = \rho_\pm(b,a)W_{\tau_{\beta(b,a)}}$$

for all $(b,a) \in U$. This completes the proof. □

5. Trace class Pseudo-Differential Operators

Theorem 5.1. *Let $\beta \in L^2(U \times U)$ be such that*

$$\int_0^\infty \int_{-\infty}^\infty |\beta(b,a,b,a)| \frac{db\,da}{a^2} < \infty.$$

Let $\sigma : U \times \{\pm\} \to S_2$ be the symbol defined in Theorem 4.1 with

$$\alpha(b,a)(x,t) = \frac{2\pi}{\sqrt{a}}\beta(b,a,x,t), \quad (b,a),(x,t) \in U.$$

Then $T_\sigma : L^2(U) \to L^2(U)$ is a trace class operator and

$$\operatorname{tr}(T_\sigma) = \int_0^\infty \int_{-\infty}^\infty \beta(b,a,b,a)\frac{db\,da}{a^2}.$$

Proof. We begin with the familiar formula

$$(T_\sigma f)(b,a) = \frac{\sqrt{a}}{2\pi}\operatorname{tr}\sum_{j=\pm}\sigma(b,a,j)D_j\hat{f}(\rho_j)\rho_j(b,a)^*), \quad (b,a) \in U,$$

for all $f \in L^2(U)$. By the same technique used in the proof of Theorem 4.1, we get

$$(T_\sigma f)(b,a) = \frac{\sqrt{a}}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(x,t)\alpha(b,a)(x,t)\frac{dx\,dt}{t^2}, \quad (b,a) \in U,$$

for all $f \in L^2(U)$. The kernel k of T_σ is of the form

$$k(b,a,x,t) = \frac{\sqrt{a}}{2\pi}\alpha(b,a)(x,t), \quad (b,a),(x,t) \in U.$$

Since

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty |k(b,a,b,a)|\frac{db\,da}{a^2} &= \int_0^\infty \int_{-\infty}^\infty \frac{\sqrt{a}}{2\pi}|\alpha(b,a)(b,a)|\frac{db\,da}{a^2} \\ &= \int_0^\infty \int_{-\infty}^\infty |\beta(b,a,b,a)|\frac{db\,da}{a^2} < \infty,\end{aligned}$$

it follows that T_σ is a trace class operator and

$$\operatorname{tr}(T_\sigma) = \int_0^\infty \int_{-\infty}^\infty \beta(b,a,b,a)\frac{db\,da}{a^2}.$$

□

6. Fourier–Wigner Transforms and Weyl Transforms

Let $(c,d) = (b,a)\cdot(b,a)$, where $(b,a)\in U$ and $\cdot$ is the binary operation in affine group. Then

$$(c,d) = (b+ab, a^2) = (c,d)$$

giving

$$a = \sqrt{d}$$

and

$$b = \frac{c}{1+\sqrt{d}}.$$

So, we can define the squre root $(c,d)^{1/2}$ of (c,d) as

$$(c,d)^{1,2} = (b,a) = \left(\frac{c}{1+\sqrt{d}}, \sqrt{d}\right).$$

Let $f,g \in L^2(U)$. Then we define the Fourier–Wigner transform $V(f,g)$ of f and g by

$$(V(f,g)(\pm,\xi))\Phi)(s) = \int_0^\infty \int_{-\infty}^\infty f(\xi^{1/2}\cdot z)\overline{g(z^{-1}\cdot\xi^{1/2})}(\rho_\pm(z)D_\pm\Phi)(s)\frac{dx\,dy}{y^2}, \quad s\in\mathbb{R}_\pm, \tag{6.1}$$

which is the same as

$$(V(f,g)(\rho_\pm,\xi)\Phi\big)(s) = (((\mathcal{F}K^\xi)(\rho_\pm)\Phi)(s), \quad s\in\mathbb{R}_\pm, \tag{6.2}$$

for all $\xi\in(b,a)\in U$, $\Phi\in L^2(\mathbb{R}_\pm)$, $z=(x,y)\in U$ and

$$K^\xi(z)f(\xi^{1/2}\cdot z)\overline{g(z^{-1}\cdot\xi^{1/2})}, \quad z\in U.$$

Let $f,g\in L^2(U)$. Then we define the Wigner transform $W(f,g)$ of f and g on $U\times\widehat{U}$ by

$$W(f,g)(z,\rho_\pm) = (\mathcal{F}_2\mathcal{F}_1^{-1}V(f,g))(z,\rho_\pm), \quad (z,\rho_\pm)\in U\times\widehat{U}, \tag{6.3}$$

where $\mathcal{F}_1^{-1}V(f,g)$ is the inverse Fourier transform of $V(f,g)$ with respect to the first variable evaluated at $z=(x,y)\in U$ and $\mathcal{F}_2V(f,g)$ is the Fourier transform of $V(f,g)$ with respect to the second variable evaluated at $\rho_\pm$. Therefore

$$(W(f,g)(z,\rho_\pm)\Phi)(s) = \int_0^\infty \int_{-\infty}^\infty f(\xi^{1/2}\cdot z)\overline{g(z^{-1}\cdot\xi^{1/2})}(\rho_\pm(\xi)D_\pm\Phi)(s)\frac{db\,da}{a^2} \tag{6.4}$$

for all $z,\xi\in U, \Phi\in L^2(\mathbb{R}_\pm)$ and $s\in\mathbb{R}_\pm$. Let $L^2(\widehat{U}\times U, S^2)$ be the space of all measurable functions $K:\widehat{U}\times U\longrightarrow S^2$ such that

$$K(\rho_\pm,z)\in S^2(L^2(\mathbb{R}_\pm))$$

and

$$\|K\|_{L^2(\widehat{U}\times U,S^2)} = \int_0^\infty\int_{-\infty}^\infty (\|K(\rho_+,z)\|^2_{S^2{}_+} + \|K(\rho_-,z)\|^2_{S^2{}_-})\frac{dx\,dy}{y^2}.$$

An inner product in $L^2(\widehat{U}\times U,S^2)$ defined by

$$(K,M)_{L^2(\hat{U}\times U,S^2)} = \int_0^\infty\int_{-\infty}^\infty\sum_{j=\pm}\mathrm{tr}(K(\rho_j,z)M(\rho_j,z)^*)\frac{dx\,dy}{y^2}. \tag{6.5}$$

for all K and M in $L^2(\hat{U} \times U, S^2)$. Similarly, let $L^2(U \times \hat{U}, S^2)$ be the space of all measurable functions $K : U \times \widehat{U} \longrightarrow S^2$ such that $K(z, \rho_\pm) \in S^2(L^2(\mathbb{R}_\pm))$ and

$$\|K\|_{L^2(U\times\widehat{U},S^2)} = \int_0^\infty \int_{-\infty}^\infty (\|K(z,\rho_+)\|^2_{S^2{}_+} + \|K(z,\rho_-)\|^2_{S^2{}_-})\frac{dx\,dy}{y^2}.$$

An inner product in $L^2(U \times \widehat{U}, S^2)$ defined by

$$(K, M)_{L^2(U\times\widehat{U},S^2)} = \int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \operatorname{tr}(K(z,\rho_j)M(z,\rho_j))\frac{dx\,dy}{y^2} \tag{6.6}$$

for all K and M in $L^2(U \times \widehat{U}, S^2)$. Let $L^2(\widehat{U}, S^2)$ be space of all meaurable functions $F : \widehat{U} \to S^2$ such that $F(\rho_\pm) \in S^2(L^2(\mathbb{R}_\pm)$. It can be seen that $L^2(\widehat{U} \times S^2)$ is a Hilbert space with inner product $(\, , \,)_{L^2(\widehat{U}\times S^2)}$ gven by

$$(F, G)_{L^2(\widehat{U},S^2)} = \operatorname{tr}(F(\rho_+)G(\rho_+)^*) + \operatorname{tr}(F(\rho_-)G(\rho_-)^*) \tag{6.7}$$

for all F and G in $L^2(\widehat{U}, S^2)$. Also, $L^2(\widehat{U}, S^2)$ is a Hilbert space in which the inner product is given by (6.7) for all $F, G \in L^2(\widehat{U}, S^2)$. Hence for all F in $L^2(\widehat{U}, S^2)$,

$$\|F\|^2_{L^2(\widehat{U},S^2)} = \|F(\rho_+)\|^2_{S^2{}_+} + \|F(\rho_-)\|^2_{S^2{}_-}. \tag{6.8}$$

Then by 2.2, we have for all f $\in L^2(U)$,

$$\|f\|^2_{L^2(U)} = \|\hat{f}\|_{L^2(\widehat{U},S^2)}.$$

Theorem 6.1. *Let $f_1, f_2, g_1, g_2 \in L^2(U)$. Then*

$$(V(f_1, g_1), V(f_2, g_2)_{L^2(\widehat{U}\times U,S^2)} = (f_1, f_2)_{L^2(U))}\overline{(g_1, g_2)_{L^2(U)}}.$$

Proof. For all $\xi = (b, a) \in U$, let K_1^ξ and K_2^ξ be defined by

$$(K_j^\xi)(z) = (f_j(\xi^{1/2} \cdot z)\overline{g_j(z^{-1} \cdot \xi^{1/2})}, \quad z \in U, j = 1, 2.$$

Then

$$\begin{aligned}
&(V(f_1, g_1), V(f_2, g_2))_{L^2(\widehat{U}\times U,S^2)} \\
=\;& \int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \operatorname{tr}(V(f_1,g_1)(\rho_j,z)V(f_2,g_2)(\rho_j,z)^*)\frac{dx\,dy}{y^2} \\
=\;& \int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \operatorname{tr}\left(\widehat{K_1^\xi}(\rho_j)\widehat{K_2^\xi}(\rho_j)^*\right)\frac{dx\,dy}{y^2} \\
=\;& \int_0^\infty \int_{-\infty}^\infty \left(\widehat{K_1^\xi}(\rho_j), \widehat{K_2^\xi}(\rho_j)\right)_{L^2(\widehat{U},S^2)} \frac{dx\,dy}{y^2} \\
=\;& \int_0^\infty \int_{-\infty}^\infty \left(K_1^\xi, K_2^\xi\right)_{L^2(U)} \frac{dx\,dy}{y^2} \\
=\;& \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f_1(\xi^{1/2}\cdot z)\overline{g_1(z^{-1}\cdot\xi^{1/2})f_2(\xi^{1/2}\cdot z)}g_2(z^{-1}\cdot\xi^{1/2})\frac{db\,da}{a^2}\frac{dx\,dy}{y^2}.
\end{aligned}$$

Let $\tilde{z} = \xi^{1/2} \cdot z$. Then because of the left invariance of the Haar measure,

$$\frac{d\tilde{x}\, d\tilde{y}}{\tilde{y}^2} = \frac{dx\, dy}{y^2}.$$

So,

$$\begin{aligned} & (V(f_1, g_1), V(f_2, g_2))_{L^2(\widehat{U}\times U, S^2)} \\ = & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f_1(\tilde{z}) \overline{g_1(\tilde{z}^{-1}\cdot \xi) f_2(\tilde{z})} g_2(\tilde{z}^{-1}\cdot\xi) \frac{d\tilde{x}\, d\tilde{y}}{\tilde{y}^2} \frac{db\, da}{a^2}. \end{aligned}$$

Let $\tilde{\xi} = \tilde{z}^{-1}\cdot\xi$. Then by the left invariance again, $\frac{d\tilde{b}\, d\tilde{a}}{\tilde{a}^2} = \frac{db\, da}{a^2}$. Therefore

$$\begin{aligned} & (V(f_1, g_1), V(f_2, g_2))_{L^2(\widehat{U}\times U, S^2)} \\ = & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f_1(\tilde{z}) \overline{g_1(\tilde{\xi}) f_2(\tilde{z})} g_2(\tilde{\xi}) \frac{d\tilde{x}\, d\tilde{y}}{\tilde{y}^2} \frac{d\tilde{b}\, d\tilde{a}}{\tilde{a}^2}. \end{aligned}$$

So,

$$(V(f_1, g_1), V(f_2, g_2))_{L^2(\widehat{U}\times U, S^2)} = (f_1, f_2)_{L^2(U)} \overline{(g_1, g_2)_{L^2(U)}}. \tag{6.9}$$

□

Theorem 6.2. *Let $f_1, f_2, g_1, g_2 \in L^2(U)$. Then*

$$(W(f_1, g_1), W(f_2, g_2))_{L^2(U\times\widehat{U}, S^2)} = (f_1, f_2)_{L^2(U)} \overline{(g_1, g_2)_{L^2(U)}}.$$

Proof We have

$$\begin{aligned} & (W(f_1, g_1), W(f_2, g_2))_{L^2(U\times\widehat{U}, S^2)} \\ = & \int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \mathrm{tr}(W(f_1, g_1)(z, \rho_j) W(f_2, g_2)(z, \rho_j)^*) \frac{dx\, dy}{y^2} \\ = & \int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \mathrm{tr}\left(\widehat{K_1^z}(\rho_j) \widehat{K_2^z}(\rho_j)^* \right) \frac{dx\, dy}{y^2} \\ = & \int_0^\infty \int_{-\infty}^\infty (\widehat{K_1^z}(\rho_j), \widehat{K_2^z}(\rho_j))_{L^2(\widehat{U}, HS)} \frac{dx\, dy}{y^2} \\ = & \int_0^\infty \int_{-\infty}^\infty (K_1^z, K_2^z)_{L^2(U)} \frac{dx\, dy}{y^2} \\ = & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f_1(\xi^{1/2}\cdot z) \overline{g_1(z^{-1}\cdot\xi^{1/2}) f_2(\xi^{1/2}\cdot z)} g_2(z^{-1}\cdot\xi^{1/2}) \frac{db\, da}{a^2} \frac{dx\, dy}{y^2}. \end{aligned}$$

Let $\tilde{z} = \xi^{1/2}\cdot z$. Then by left invariance, $\frac{d\tilde{x}\, d\tilde{y}}{\tilde{y}^2} = \frac{dx\, dy}{y^2}$. Therefore

$$\begin{aligned} & (W(f_1, g_1), W(f_2, g_2)_{L^2(U\times\widehat{U}, S^2)} \\ = & \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f_1(\tilde{z} \overline{g_1(\tilde{z}^{-1}\cdot\xi) f_2(\tilde{z})} g_2(\tilde{z}^{-1}\cdot\xi) \frac{d\tilde{x}\, d\tilde{y}}{\tilde{y}^2} \frac{db\, da}{a^2}. \end{aligned}$$

Let $\tilde{\xi} = \tilde{z}^{-1} \cdot \xi$. Then by left invariance again, $\frac{d\tilde{b}\, d\tilde{a}}{\tilde{a}^2} = \frac{db\, da}{a^2}$. Therefore

$$\begin{aligned}
&(W(f_1, g_1), W(f_2, g_2))_{L^2(U\times\widehat{U}, S^2)} \\
&= \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f_1(\tilde{z}) \overline{g_1(\tilde{\xi}) f_2(\tilde{z})} g_2(\tilde{\xi}) \frac{d\tilde{x}\, d\tilde{y}}{\tilde{y}^2} \frac{d\tilde{b}\, d\tilde{a}}{\tilde{a}^2} \\
&= (f_1, f_2)_{L^2(U)} \overline{(g_1, g_2)_{L^2(U)}}.
\end{aligned}$$

and the proof is complete. □

Let $\sigma : U \times \{\pm\} \to S^2$ be an operator-valued symbol. Then we define the Weyl transform W_σ associated to the symbol σ by

$$(W_\sigma f, g)_{L^2(U)} = \int_0^\infty \int_{-\infty}^\infty \left[\sum_{j=\pm} \mathrm{tr}(\sigma(b, a, \rho_j) D_j W(f, g)(b, a, \rho_j)) \right] \frac{db\, da}{a^2} \tag{6.10}$$

for all f and g in $L^2(U)$.

Theorem 6.3. *Let $\sigma : U \times \{\pm\} \to S^2$ be an operator-valued symbol such that the mappings*

$$U \times \{\pm\} \ni (b, a, j) \mapsto D_j \sigma(b, a, \rho_j)^* \in S_2(\mathbb{R}_j)$$

with $j \in \{\pm\}$ are in $L^2(U \times \hat{U}, S^2)$. Then $W_\sigma : L^2(U) \to L^2(U)$ is a bounded linear operator.

Proof. Let $f, g \in L^2(U)$. Then by (6.10), the Schwarz inequality and the Moyal identity for the Wigner transforms,

$$\begin{aligned}
|(W_\sigma f, g)_{L^2(U)}| &\le \int_0^\infty \int_{-\infty}^\infty \left[\sum_{j=\pm} |\mathrm{tr}(\sigma(b, a, \rho_j) D_j W(f, g)(b, a, \rho_j))| \right] \frac{db\, da}{a^2} \\
&= \sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty |\mathrm{tr}(\sigma(b, a, \rho_j) D_j W(f, g)(b, a, \rho_j))| \frac{db\, da}{a^2} \\
&\le \sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b, a, \rho_j) D_j\|_{S^2} \|W(f, g)(b, a, \rho_j)\|_{S^2} \frac{db\, da}{a^2} \\
&\le \sum_{j=\pm} \left(\int_0^\infty \int_{-\infty}^\infty \|\sigma(b, a, \rho_j) D_j\|_{S^2}^2 \frac{db\, da}{a^2} \right)^{1/2} \times \\
&\quad \left(\int_0^\infty \int_{-\infty}^\infty \|W(f, g)(b, a, \rho_j)\|_{S^2}^2 \frac{db\, da}{a^2} \right)^{1/2} \\
&\le \left(\int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \|\sigma(b, a, \rho_j) D_j\|_{S^2}^2 \frac{db\, da}{a^2} \right)^{1/2} \times \\
&\quad \left(\int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \|W(f, g)(b, a, \rho_j)\|_{S^2}^2 \frac{db\, da}{a^2} \right)^{1/2} \\
&\le \left(\int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \|(\sigma(b, a, \rho_j) D_j)^*\|_{S^2}^2 \frac{db\, da}{a^2} \right)^{1/2} \times
\end{aligned}$$

$$\left(\int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \|W(f,g)(b,a,\rho_j)\|_{S^2}^2 \frac{db\,da}{a^2}\right)^{1/2}$$
$$\leq \left(\int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \|D_j\sigma(b,a,\rho_j)^*\|_{S^2}^2 \frac{db\,da}{a^2}\right)^{1/2} \times$$
$$\left(\int_0^\infty \int_{-\infty}^\infty \sum_{j=\pm} \|W(f,g)(b,a,\rho_j)\|_{S^2}^2 \frac{db\,da}{a^2}\right)^{1/2}$$
$$\leq \|D\sigma(b,a,\rho)^*\|_{L^2(U\times\widehat{U},S^2)} \|W(f,g)\|_{L^2(U\times\widehat{U},S^2)}$$
$$\leq \|D\sigma(b,a,\rho)^*\|_{L^2(U\times\widehat{U},S^2)} \|f\|_{L^2(U)} \|g\|_{L^2(U)}.$$

□

References

[1] Dasgupta, A. and Wong, M. W., Pseudo-Differential Operators on the Affine Group, in *Pseudo-Differential Operators: Groups, Geometry and Applications*, Trends in Mathematics, Birkhäuser, (2017), 1–14.

[2] Dasgupta, A. and Wong, M. W., Hilbert–Schmidt and trace class pseudo-differential operators on the Heisenberg group", *Pseudo-Differ. Oper. Appl.* **4** (2013), 345–359.

[3] Dasgupta, A. and Wong, M. W., Weyl transforms for H-type groups, *J. Pseudo-Differ. Oper. Appl.* **6** (2015), 11–19

[4] Duflo, M. and Moore, C.C., On the regular representation of a non-unimodular locally compact group, *J. Funct. Anal.*, **21** (1976), 209–243.

[5] Molahajloo, S. and Wong, K. L., Pseudo-differential operators on finite abelian groups, *J. Pseudo-Differ. Oper. Appl.* **6** (2015), 1–9.

[6] Molahajloo, S. and Wong, M. W., Pseudo-differential operators on $\mathbb{S}^1$, in *New Developments in Pseudo-Differential Operators Operator Theory: Advances and Applications* **189**, Birkhäuser, 2009, 297–306.

[7] Teufel, S., *Adiabatic Perturbation Theory in Quantum Dynamics*, Springer, 2003.

[8] Wong, M. W., *Weyl Transforms*, Springer, 1998.

[9] Wong, M. W., *Wavelet Transforms and Localization Operators*, Birkhäuser, 2002.

[10] Wong, M. W., *Complex Analysis*, World Scientific, 2008.

[11] Wong, M. W., *An Introduction to Pseudo-Differential Operators*, Third Edition, World Scientific, 2014.

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