Normality, Self-Adjointness, Spectral Invariance, Groups and Determinants of Pseudo-Differential Operators on Finite Abelian Groups

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Abstract. We give the normality, self-adjointness and spectral invariance of pseudo-differential operators on finite abelian groups. We also give a formula for the determinant of every element in a group of pseudo-differential operators on a finite abelian group.

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1. Introduction

In [2, 5, 6], pseudo-differential operators on the additive group $\mathbb{Z}_N$ related to modular arithmetic are motivated by discretizations of pseudo-differential operators on the unit circle $\mathbb{S}^1$ centered at the origin in the complex plane, which have recently been studied in [1, 8, 9]. In [7], the basic theory of pseudo-differential operators on finite abelian groups is developed. In particular, the formulas for the products and the adjoints of pseudo-differential operators are given. Of particular interest in [7] is the solution of the spectral invariance problem for pseudo-differential operators on a finite abelian group of order 2. We give in this paper two proofs of the full solution to the spectral invariance problem for pseudo-differential operators on a finite abelian group of arbitrary order. In explicit terms, we give two proofs of

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the fact that if a pseudo-differential operator on a finite abelian group is invertible, then its inverse is also a pseudo-differential operator. Results on groups of pseudo-differential operators on finite abelian groups are also presented.

We recall in Section 2 the Fourier analysis on finite abelian groups given in [7]. Products and adjoints of pseudo-differential operators on finite abelian groups recalled in Section 3 are the essential results we need in this paper. To supplement the results in Section 3, we first prove in Section 4 that the mapping of symbols to pseudo-differential operators is injective. Then in Section 5 we use the product formula and the adjoint formula in Section 3 to give a criterion on the normality of pseudo-differential operators on finite abelian groups. In particular, a criterion on the self-adjointness, or equivalently, non-self-adjointness, is also given. This is particularly timely in view of the current trends in the study of non-self-adjoint operators. Useful sufficient condtions on self-adjointness of pseudo-differential opertors on finite abelian groups are also given. The main result on spectral invariance is given and proved in Section 6. This also follows from the universality theorem that we prove in Section 7. We conclude in Section 8 with results on groups of pseudo-differential operators on finite abelian groups.

2. Fourier Analysis on Finite Abelian Groups

Let G be a finite abelian group of order $|G|$ equipped with the binary operation $G \times G \ni (g,h) \mapsto gh \in G$. We denote by g_1 the identity element in G. A complex-valued function $\chi : G \to \mathbb{S}^1$ such that

$$\chi(gh) = \chi(g)\chi(h), \quad g, h \in G,$$

is called a *character* of G. Let $\widehat{G}$ be the set of all characters of G. Then $\widehat{G}$ becomes an abelian group when it is equipped with the group law given by

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g), \quad g \in G,$$

for all characters χ_1 and χ_2 of G. The identity element in $\widehat{G}$ is denoted by χ_1.

Let $L^2(G)$ be the set of all complex-valued functions on G. Then it is easy to see that $L^2(G)$ is the same as $\mathbb{C}^n$, where $n = |G|$ in this paper. The inner product $(\, , \,)$ and the norm $\| \, \|$ in $L^2(G)$ are defined by

$$(u, v) = \frac{1}{|G|} \sum_{g \in G} u(g)\overline{v(g)}$$

and

$$\|u\|^2 = \frac{1}{|G|} \sum_{g \in G} |u(g)|^2$$

for all u and v in $L^2(G)$. Of fundamental importance in this paper is the following well-known theorem.

Theorem 2.1. *The characters of G form an orthonormal basis for $L^2(G)$.*

Let $u \in L^2(G)$. Then the *Fourier transform* $\hat{u}$ of u is the function on $\widehat{G}$ defined by

$$\hat{u}(\chi) = (u, \chi) = \frac{1}{|G|} \sum_{g \in G} u(g)\overline{\chi(g)}, \quad \chi \in \widehat{G}.$$

The most basic result is the *Plancherel formula* to the effect that

$$\|u\|^2 = \sum_{\chi \in \widehat{G}} |\hat{u}(\chi)|^2, \quad u \in L^2(G).$$

From the Plancherel formula, we obtain the following *Fourier inversion formula.*

Theorem 2.2. *Let $u \in L^2(G)$. Then*

$$u = \sum_{\chi \in \widehat{G}} \hat{u}(\chi)\chi.$$

If we write $G = \{g_1, g_2, \ldots, g_n\}$ and $\widehat{G} = \{\chi_1, \chi_2, \ldots, \chi_n\}$, then the Fourier inversion formula can be nailed down to

$$u = \sum_{k=1}^{n} \hat{u}(\chi_k)\chi_k, \quad u \in L^2(G).$$

More details can be found in [10, 11, 12]. The book [13] contains details on the Fourier analysis on $\mathbb{Z}_N$.

The Fourier inversion formula is a formula for the identity operator on $L^2(G)$ based on the Fourier transform. The identity operator has perfect symmetry and this perfect symmetry renders it uninteresting. We need to break the symmetry in order to obtain more useful operators known as pseudo-differential operators by inserting symbols into the Fourier inversion formula. Symbols in this paper are just functions on $G \times \widehat{G}$.

3. Pseudo-Differential Operators on Finite Abelian Groups

Let σ be a function on the phase space $G \times \widehat{G}$. Then we define the *pseudo-differential operator* T_σ on G corresponding to the *symbol* σ by

$$(T_\sigma u)(g) = \sum_{\chi \in \widehat{G}} \sigma(g, \chi)\hat{u}(\chi)\chi(g), \quad g \in G.$$

In order to compute the matrix $[T_\sigma]_\chi$ corresponding to the orthonormal basis $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ for $L^2(G)$, we just note that

$$[T_\sigma]_\chi = [\sigma_{lk}]_{1 \le l,k \le |G|},$$

where σ_{lk} is the entry of the matrix in the l^{th} row and k^{th} column. Indeed,

$$\sigma_{lk} = (T_\sigma \chi_k, \chi_l) = (\sigma(\cdot, \chi_k)\chi_k, \chi_l).$$

Thus,

$$[T_\sigma]_\chi = \begin{bmatrix} (\sigma(\cdot,\chi_1)\chi_1,\chi_1) & (\sigma(\cdot,\chi_2)\chi_2,\chi_1) & \cdots & (\sigma(\cdot,\chi_n)\chi_n,\chi_1) \\ (\sigma(\cdot,\chi_1)\chi_1,\chi_2) & (\sigma(\cdot,\chi_2)\chi_2,\chi_2) & \cdots & (\sigma(\cdot,\chi_n)\chi_n,\chi_2) \\ \vdots & \vdots & \vdots & \vdots \\ (\sigma(\cdot,\chi_1)\chi_1,\chi_n) & (\sigma(\cdot,\chi_2)\chi_2,\chi_n) & \cdots & (\sigma(\cdot,\chi_n)\chi_n,\chi_n) \end{bmatrix}. \quad (3.1)$$

We have the following theorems on the products and adjoints of pseudo-differential operators on finite abelian groups given in [7].

Theorem 3.1. *Let σ and τ be functions on $G \times \widehat{G}$. Then*

$$[T_\sigma]_\chi [T_\tau]_\chi = [T_\lambda]_\chi,$$

where

$$\lambda_{lk} = \sum_{m=1}^{n} (\sigma(\cdot,\chi_m)\chi_m,\chi_l)(\tau(\cdot,\chi_k)\chi_k,\chi_m)$$

for $1 \le k, l \le n$.

Theorem 3.2. *Let σ be a function on $G \times \widehat{G}$. Then*

$$[T_\sigma^*]_\chi = [T_\tau]_\chi,$$

where

$$\tau_{lk} = \overline{\sigma_{kl}}$$

for $1 \le l, k \le n$.

We also need the trace formula in [7].

Theorem 3.3. *Let $\sigma : G \times \widehat{G} \to \mathbb{C}$ be a symbol. Then the trace $\mathrm{tr}([T_\sigma]_\chi)$ of the matrix $[T_\sigma]_\chi$ is given by*

$$\mathrm{tr}\left([T_\sigma]_\chi\right) = \frac{1}{|G|} \sum_{j=1}^{|G|} \sum_{k=1}^{|G|} \sigma(g_j,\chi_k).$$

4. Injectivity

We first prove that the mapping of symbols to pseudo-differential operators on finite abelian groups is injective.

Theorem 4.1. *Let σ and τ be complex-valued functions on $G \times \widehat{G}$ such that $T_\sigma = T_\tau$. Then $\sigma = \tau$.*

Proof By (3.1), we get for $k, l = 1, 2, \ldots, |G|$,

$$(\sigma(\cdot,\chi_k),\chi_l) = (\tau(\cdot,\chi_k)\chi_k,\chi_l).$$

Therefore for $k = 1, 2, \ldots, |G|$,

$$\sigma(\cdot,\chi_k)\chi_k = \tau(\cdot,\chi_k)\chi_k.$$

Thus, if we multiply both sides of the preceding equation by $\overline{\chi_k}$, we get for $k = 1, 2, \ldots, |G|$,

$$\sigma(\cdot, \chi_k)|\chi_k|^2 = (\tau(\cdot, \chi_k)|\chi_k|^2.$$

Since χ_k is a character, it follows that $|\chi_k| = 1$, $k = 1, 2, \ldots, |G|$. Therefore

$$\sigma(\cdot, \chi_k) = \tau(\cdot, \chi_k), \quad k = 1, 2, \ldots, |G|.$$

So, $\sigma = \tau$ and the proof is complete. □

5. Normal and Self-Adjoint Operators

The product formula and the adjoint formula in Section 3 give us immediately a criterion for a pseudo-differential operator on a finite abelian group to be normal.

Theorem 5.1. *Let σ be a function on $G \times \widehat{G}$. Then the pseudo-differential operator T_σ is normal if and only if for all $k, l = 1, 2, \ldots, |G|$,*

$$\sum_{j=1}^{|G|} (\sigma_{lj}\overline{\sigma_{kj}} - \overline{\sigma_{jl}}\sigma_{jk}) = 0,$$

or more explicitly,

$$\sum_{j=1}^{|G|} [(\sigma(\cdot, \chi_j)\chi_j, \chi_l)\overline{(\sigma(\cdot, \chi_j)\chi_j, \chi_k)} - \overline{(\sigma(\cdot, \chi_l)\chi_l, \chi_j)}(\sigma(\cdot, \chi_k)\chi_k, \chi_j)] = 0.$$

It follows from Theorem 5.1 that if σ is a symbol independent of $g \in G$ or independent of $\chi \in \widehat{G}$, then T_σ is normal.

Let us observe here that a criterion for self-adjointness of pseudo-differential operators on finite abelian groups can be obtained.

Theorem 5.2. *Let σ be a function on $G \times \widehat{G}$, where G is a finite abelian group. Then the pseudo-differential operator on G is self-adjoint if and only if for all $k, l = 1, 2, \ldots, |G|$,*

$$(\chi_k, \overline{\sigma(\cdot, \chi_k)}\chi_l) = (\chi_k, \sigma(\cdot, \chi_l)\chi_l).$$

We note that if σ is a real-valued symbol independent of $g \in G$ or independent of $\chi \in \widehat{G}$, then T_σ is self-adjoint.

6. A Spectrally Invariant C^*-Algebra

We denote by $\mathcal{A}$ the set of all matrices $[T_\sigma]_\chi$, where $\sigma : G \times \widehat{G} \to \mathbb{C}$ is a symbol. It is clear from Theorems 3.1 and 3.2 that $\mathcal{A}$ is a C^*-algebra. It is well known that the rank-nullity theorem in linear algebra [4] tells us that every pseudo-differential operator in $\mathcal{A}$ as a linear operator from $L^2(G)$ into $L^2(G)$ is Fredholm with zero index. Hence for every symbol $\sigma : G \times \widehat{G} \to \mathbb{C}$, the pseudo-differential operator on

G is invertible if and only if $\det [T_\sigma]_\chi \neq 0$. We have the following theorem on the spectral invariance of the C^*-algebra $\mathcal{A}$.

Theorem 6.1. *Let $A \in \mathcal{A}$ be an invertible matrix. Then $A^{-1} \in \mathcal{A}$.*

Proof Since $A \in \mathcal{A}$, it follows from Theorema 3.1 and 3.2 that A^*A is in $\mathcal{A}$. So, for every positive number t,

$$e^{-tA^*A} = \sum_{k=0}^{\infty} \frac{(-1)^k t^k (A^*A)^k}{k!},$$

where the convergence is in the operator norm of $\mathcal{A}$. Since $\mathcal{A}$ is a C^*-algebra, it follows that

$$e^{-tA^*A} \in \mathcal{A}.$$

Thus,

$$(A^*A)^{-1} = \int_0^\infty e^{-tA^*A} dt \in \mathcal{A}.$$

So, $A^{-1}(A^*)^{-1} \in \mathcal{A}$ and hence

$$A^{-1} = (A^{-1}(A^*)^{-1})A^* \in \mathcal{A}.$$

□

7. Universality

The spectral inversion theorem in the previous section follows from the following universality theorem.

Theorem 7.1. *Every bounded linear operator on $L^p(G)$, $1 \leq p \leq \infty$, is a pseudo-differential operator from $L^p(G)$ into $L^p(G)$.*

Proof Let $u \in L^p(G)$. Then for all $\chi \in \widehat{G}$, we get by the Fourier inversion formula

$$\begin{aligned}
\widehat{Au}(\chi) &= \frac{1}{|G|}\sum_{g\in G}(Au)(g)\overline{\chi(g)} \\
&= \frac{1}{|G|}\sum_{g\in G} u(g)\overline{(A^*\chi)(g)} \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\omega\in\widehat{G}} \hat{u}(\omega)\omega(g)\overline{(A^*\chi)(g)} \\
&= \frac{1}{|G|}\sum_{\omega\in\widehat{G}} \hat{u}(\omega)\sum_{g\in G}\omega(g)\overline{(A^*\chi)(g)} \\
&= \frac{1}{|G|}\sum_{\omega\in\widehat{G}}\sum_{g\in G}(A\omega)(g)\overline{\chi(g)}\hat{u}(\omega)\omega(g) \\
&= \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\sum_{\omega\in\widehat{G}}(A\omega)(g)\hat{u}(\omega)\omega(g).
\end{aligned}$$

Therefore

$$(Au)(g) = \sum_{\omega\in\widehat{G}}(A\omega)(g)\hat{u}(\omega)\omega(g) = \sum_{\omega\in\widehat{G}}\sigma(g,\omega)\hat{u}(\omega)\omega(g) = (T_\sigma u)(g)$$

for all $g \in G$, where

$$\sigma(g,\omega) = \frac{1}{|G|}\sum_{g\in G}(A\omega)(g), \quad (g,\omega) \in G \times \widehat{G}.$$

□

8. Groups

In view of the spectral invariance given in Theorem 6.1, the biggest group of pseudo-differential operators sitting inside $\mathcal{A}$ is the group $G(\mathcal{A})$ given by

$$G(\mathcal{A}) = \{A \in \mathcal{A} : \det A \neq 0\}$$

and the group law is multiplication of matrices. We can now give a subgroup of $G(\mathcal{A})$ in which the determinant of every pseudo-differential operator can be explicitly computed. Let $e^{\mathcal{A}}$ be the subset of $G(\mathcal{A})$ defined by

$$e^{\mathcal{A}} = \{e^A : A \in \mathcal{A}\}.$$

Then we have the following theorem.

Theorem 8.1. *$e^{\mathcal{A}}$ is a group with respect to matrix multiplication.*

Proof Let A and B be elements in $\mathcal{A}$. Then by the Baker–Campbell–Hausdorff formula in, e.g., [3],

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]},$$

where $[A,B]$ is the commutator of A and B. Thus, using Theorem 3.1 to the effect that $\mathcal{A}$ is an algebra, $e^A e^B \in \mathcal{A}$. It is obvious that the identity matrix of order $n \times n$ is the identity element of $e^{\mathcal{A}}$. Let $A \in \mathcal{A}$. Then e^{-A} is the inverse of e^A. Therefore $e^{\mathcal{A}}$ is a group with respect to matrix multiplication. □

9. Determinants

Let $A \in \mathcal{A}$. Then using the Leibniz formula for the determinants of matrices and (3.1), we have for all symols $\sigma \in G \times \widehat{G}$,

$$\det([T_\sigma]_\chi) = \sum_{p \in S_{|G|}} \mathrm{sig}\,(p) \prod_{j=1}^{|G|} (\sigma(\cdot, \chi_{p(j)})\chi_{p(j)}, \chi_j),$$

where the summation is taken over all permutations p of the symmetric group of the integers $1, 2, \ldots, |G|$, and $\mathrm{sig}\,(p)$ is the signature of p.

We can now give a result on the determinant of every element in $e^{\mathcal{A}}$.

Theorem 9.1. *Let $\sigma : G \times \widehat{G} \to \mathbb{C}$ be a symbol. Then*

$$\det(e^{[T_\sigma]_\chi}) = \exp\left(\frac{1}{|G|}\sum_{j=1}^{|G|}\sum_{k=1}^{|G|} \sigma(g_j, \chi_k)\right).$$

Proof We use the well-known formula stating that for all $n \times n$ matrices,

$$\det(e^A) = e^{\mathrm{tr}\,(A)}.$$

Therefore using Theorem 3.3,

$$\det(e^{[T_\sigma]_\chi}) = \exp\left(\sum_{j=1}^{|G|}\sum_{k=1}^{|G|} \sigma(g_j, \chi_k)\right).$$

□

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