Pseudo-Differential Analysis of Bounded Linear Operators from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$

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Abstract: In this paper, we first show that every bounded linear operator T from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ for $1 < p_1, p_2 < \infty$ is a pseudo-differential operator and give a formula for its symbol σ. Using the pseudo-differential representation of T, we offer necessary and sufficient conditions on the symbols σ to find the symbol of the adjoint T^*. We present necessary and sufficient conditions on the symbols σ to ensure that their corresponding bounded linear operators from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ are self-adjoint, compact or compact self-adjoint. As applications, in case σ is a real-valued function, we show that the corresponding bounded linear operator T is self-adjoint if and only if σ depends only on one variable. Also, we prove that every compact operator from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ is the product of two compact operators and give some formulas for the symbols of compact and compact self-adjoint operators.

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1 Introduction

Let $\mathbb{S}^1$ be the unit circle centered at the origin and let $\mathbb{Z}$ be the set of all integers. For every measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ and every measurable function f on $\mathbb{S}^1$, we define the function $T_\sigma f$ on $\mathbb{S}^1$ *formally* by

$$(T_\sigma f)(\theta) = \sum_{n\in\mathbb{Z}} e^{in\theta}\sigma(\theta, n)\hat{f}(n), \quad \theta \in [-\pi, \pi],$$

where $\hat{f}$ is the Fourier transform of f given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

We call T_σ the pseudo-differential operator on $\mathbb{S}^1$ with symbol σ. Suitable conditions on σ can lead to boundedness, compactness, Fredholmness, nuclearity and self-adjointness of the corresponding pseudo-differential operator T_σ [1, 2, 11, 12, 13, 14]. In [6], it is shown that if $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ with symbol σ in the Hörmander class $S^m_{1,0}(\mathbb{S}^1 \times \mathbb{Z})$ is a bounded pseudo-differential operator, then

$$\sigma^*(\theta, n) = \sum_{m\in\mathbb{Z}} e^{im\theta}\overline{\widehat{g_{m+n}}(-m)}$$

for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$, where σ^* is the symbol of T^*_σ and

$$g_n(\theta) = \sigma(\theta, n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

In [5], it is proved that if $\sum_{n=-\infty}^{\infty} \sigma(\theta, n)$ is an absolutely convergent series and $\sup_{\theta\in[-\pi,\pi]} |\sigma(\theta, n)| < \infty$, then the pseudo-differential operator $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$, $1 < p < \infty$, is a compact operator. in [8], it is proved that $T_\sigma : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 \le p_1, p_2 < \infty$, is nuclear if and only if there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ and $\{h_k\}_{k=-\infty}^{\infty}$ in, respectively, $L^{p'_1}(\mathbb{S}^1)$ and $L^{p_2}(\mathbb{S}^1)$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{S}^1)} \|g_k\|_{L^{p'_1}(\mathbb{S}^1)} < \infty$$

and

$$\sigma(\theta, n) = 2\pi e^{-in\theta} \sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

We have shown that $T_\sigma^* : L^{p_2'}(\mathbb{S}^1) \to L^{p_1'}(\mathbb{S}^1)$ is also nuclear and the symbol σ^* of T_σ^* is given by

$$\sigma^*(\theta, n) = 2\pi e^{in\theta} \sum_{m=-\infty}^{\infty} \overline{g_m}(\theta) \widehat{\overline{h_m}}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

The results for $\mathbb{S}^1$ can be extended easily to the n-dimensional torus $\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}$. Extensions to compact Lie groups and to compact and Hausdorff groups can be found, for instance, in [4, 7, 9, 10].

The starting point of this paper is that every bounded linear operator T from $L^{p_1}(\mathbb{S}^1)$ to $L^{p_2}(\mathbb{S}^1)$ for $1 < p_1, p_2 < \infty$ can be written as a pseudo-differential operator. We use the pseudo-differential realization of T to give necessary and sufficient conditions on σ to ensure that a measurable function τ on $\mathbb{S}^1 \times \mathbb{Z}$ is the symbol of T^*, the adjoint of T. We present necessary and sufficient conditions on the symbols σ to ensure that the corresponding bounded linear operators from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ are self-adjoint, compact or compact self-adjoint. As applications, in case σ is a real-valued function, we show that the corresponding bounded linear operator T is self-adjoint if and only if σ depends only on one variable, θ or n, but not both. Also, we show that every compact operator from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ is the product of two compact operators and give formulas for the symbols of compact and compact self-adjoint operators.

We first prove in Section 2 that every bounded linear operator T from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ for $1 < p_1, p_2 < \infty$ is a pseudo-differential operator and give a formula for its symbol σ. We show that the symbol σ is in $L^{p_2}(\mathbb{S}^1)$ and also the bounded linear operators P and T are equal if and only if their symbols are equal. In Section 3, we give necessary and sufficient conditions on a measurable function τ on $\mathbb{S}^1 \times \mathbb{Z}$ to ensure that it is the symbol of T^*. As applications, necessary and sufficient conditions are given for bounded linear operators from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ to be self-adjoint. In case σ is a real-valued function, we show that the corresponding bounded linear operator T is self-adjoint if and only if σ depends only on one variable. Also, we prove that if a measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ given by

$$\sigma(\theta, n) = \sigma_1(n)\sigma_2(\theta), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z},$$

is the symbol of a bounded linear operator $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, then T is self-adjoint if and only if σ_1 is a real constant and σ_2 is a real-valued function. In Section 4, we give necessary and sufficient conditions to guarantee that bounded linear operators $T : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1),\ 1 < p < \infty$, have eigenvalues and eigenfunctions. Necessary and sufficient conditions are given for bounded linear operators from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ to be compact or compact self-adjoint. We also give some formulas for the symbols of compact and compact self-adjoint operators. We show in Section 5 that every compact operator from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ is the product of two compact operators

2 Bounded Linear Operators

In order to show that every bounded linear operator $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, is a pseudo-differential operator, we use the following theorem in [3, 15].

Theorem 2.1 *Let $f \in L^p(\mathbb{S}^1),\ 1 < p < \infty$. Then the Fourier series of f converges to f in $L^p(\mathbb{S}^1)$.*

In the next theorem, we prove that every bounded linear operator $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, is a pseudo-differential operator and give a formula for its symbol.

Theorem 2.2 *Every bounded linear operator $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, is a pseudo-differential operator with symbol σ given by*

$$\sigma(\theta, n) = e^{-in\theta}(Te_n)(\theta), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z},$$

where $e_n(\theta) = e^{in\theta}$ for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$.

Proof Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, be a bounded linear operator. Using Theorem 2.1, for all $f \in L^{p_1}(\mathbb{S}^1)$, we have

$$\begin{aligned} (Tf)(\theta) &= T\left(\sum_{m=-\infty}^{\infty} e_m \hat{f}(m)\right)(\theta) \\ &= \sum_{m=-\infty}^{\infty} (Te_m)(\theta)\hat{f}(m) \\ &= \sum_{m=-\infty}^{\infty} e^{im\theta}\sigma(\theta, m)\hat{f}(m), \end{aligned}$$

where

$$\sigma(\theta,n) = e^{-in\theta}(Te_n)(\theta), \quad (\theta,n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

□

Remark 2.3 While Theorem 2.2 states that every bounded linear operator from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ is a pseudo-differential operator for $1 < p_1, p_2 < \infty$, it is well known that not every pseudo-differential operator even from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ is a bounded linear operator.

In the following theorem, we show that two bounded linear operators P and T are equal if and only if their corresponding symbols are equal.

Theorem 2.4 *Let $P, T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, be two bounded linear operators. Then $T = P$ if and only if for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,*

$$\sigma(\theta,n) = \eta(\theta,n),$$

where σ is the symbol of T and η is the symbol of P.

Proof Assume that $P = T$. Let σ be the symbol of T and η be the symbol of P. Then for all $f \in L^{p_1}(\mathbb{S}^1)$,

$$\begin{aligned}(Tf)(\theta) &= \sum_{m=-\infty}^{\infty} e^{im\theta}\sigma(\theta,m)\hat{f}(m) \\ &= \sum_{m=-\infty}^{\infty} e^{im\theta}\eta(\theta,m)\hat{f}(m) = (Pf)(\theta), \quad \theta \in [-\pi,\pi].\end{aligned}$$

Now, let $n \in \mathbb{Z}$ and $f = e_n$. Since

$$\widehat{e_n}(m) = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases} \tag{2.1}$$

it follows that

$$\sigma(\theta,n) = \eta(\theta,n), \quad (\theta,n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

Conversely, if σ and η are, respectively, the symbols of T and P such that

$$\sigma(\theta,n) = \eta(\theta,n), \quad (\theta,n) \in \mathbb{S}^1 \times \mathbb{Z},$$

then for all $f \in L^{p_1}(\mathbb{S}^1)$,

$$\sum_{m=-\infty}^{\infty} e^{im\theta}\sigma(\theta,m)\hat{f}(m) = \sum_{m=-\infty}^{\infty} e^{im\theta}\eta(\theta,m)\hat{f}(m), \quad \theta \in [-\pi,\pi],$$

which means that $T = P$. □

In the next theorem, we show that if $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, is a bounded linear operator with symbol σ, then σ is in $L^{p_2}(\mathbb{S}^1)$ for all $n \in \mathbb{Z}$.

Theorem 2.5 *Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, be a bounded linear operator with symbol σ. Then for all $n \in \mathbb{Z}$,*

$$\int_{-\pi}^{\pi} |\sigma(\theta,n)|^{p_2} d\theta < \infty.$$

Proof Assume that $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, is a bounded linear operator. Then for all $f \in L^{p_1}(\mathbb{S}^1)$, there exists a positive constant C such that

$$\begin{aligned}\left(\int_{-\pi}^{\pi} |(Tf)(\theta)|^{p_2} d\theta\right)^{1/p_2} &= \left(\int_{-\pi}^{\pi} \left|\sum_{m=-\infty}^{\infty} e^{im\theta}\sigma(\theta,m)\hat{f}(m)\right|^{p_2} d\theta\right)^{1/p_2} \\ &\le C\left(\int_{-\pi}^{\pi} |f(\theta)|^{p_1} d\theta\right)^{1/p_1}.\end{aligned}$$

Now, let $n \in \mathbb{Z}$ and $f = e_n$. Then by (2.1),

$$\begin{aligned}&\left(\int_{-\pi}^{\pi} |\sigma(\theta,n)|^{p_2} d\theta\right)^{1/p_2} \\ =\ &\left(\int_{-\pi}^{\pi} \left|\sum_{m=-\infty}^{\infty} e^{im\theta}\sigma(\theta,m)\widehat{e_n}(m)\right|^{p_2} d\theta\right)^{1/p_2} \\ \le\ &C\left(\int_{-\pi}^{\pi} |e^{in\theta}|^{p_1} d\theta\right)^{1/p_1} < \infty.\end{aligned}$$

□

3 Adjoints and Self-Adjointness

In this section, we give necessary and sufficient conditions on a measurable function τ on $\mathbb{S}^1 \times \mathbb{Z}$ to ensure that it is the symbol of the adjoint of a bounded linear operator $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ with symbol σ. As applications, necessary and sufficient conditions are given for bounded linear operators from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ to be self-adjoint. In case σ is a real-valued function, we show that the corresponding bounded linear operator T is self-adjoint if and only if σ depends only on one variable. Also, we prove that if a measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ given by

$$\sigma(\theta, n) = \sigma_1(n)\sigma_2(\theta), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z},$$

is the symbol of a bounded linear operator $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, then T is self-adjoint if and only if σ_1 is a real constant and σ_2 is a real-valued function. We begin with the following theorem.

Theorem 3.1 *Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, be a bounded linear operator with symbol σ. Let τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that*

$$\tau(\cdot, n) \in L^{p_1'}(\mathbb{S}^1), \quad n \in \mathbb{Z}.$$

Then τ is the symbol of $T^* : L^{p_2'}(\mathbb{S}^1) \to L^{p_1'}(\mathbb{S}^1)$ if and only if for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = \int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)}e^{in\theta}d\theta.$$

Proof Suppose that $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1,\ p_2 < \infty$, is a bounded linear operator, $T^* : L^{p_2'}(\mathbb{S}^1) \to L^{p_1'}(\mathbb{S}^1)$ is the adjoint of T and τ is the symbol of T^*. Then

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta &= \int_{-\pi}^{\pi} e^{-im\theta}(Te_n)(\theta)d\theta \\ &= \int_{-\pi}^{\pi} \overline{(T^*e_m)(\theta)}e^{in\theta}d\theta \\ &= \int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)}e^{in\theta}d\theta\end{aligned}$$

for all $n, m \in \mathbb{Z}$. Conversely, suppose that for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = \int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)}e^{in\theta}d\theta.$$

Then for all $n, m \in \mathbb{Z}$,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta,m)}e^{in\theta}d\theta &= \int_{-\pi}^{\pi} e_{-m}(\theta)(Te_n)(\theta)\,d\theta \\ &= \int_{-\pi}^{\pi} \overline{(T^*e_m)(\theta)}e_n(\theta)\,d\theta,\end{aligned}$$

which implies that

$$T^*e_m = \tau(\cdot,m)e_m, \quad m \in \mathbb{Z}.$$

Therefore τ is the symbol of $T^* : L^{p_2'}(\mathbb{S}^1) \to L^{p_1'}(\mathbb{S}^1)$. □

As a consequence of Theorem 3.1, we have the following formula for the symbol of the adjoint of a bounded linear operator $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ with symbol σ, where $1 < p_1, p_2 < \infty$.

Theorem 3.2 *Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, be a bounded linear operator with symbol σ. Let τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then τ is the symbol of T^* if and only if for every $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,*

$$\tau(\theta,n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\sigma(\theta',k+n)}d\theta'.$$

Proof Suppose that τ is the symbol of T^*. Using Theorem 3.1, we get for all $k, n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-ik\theta'}\tau(\theta',n)e^{in\theta'}d\theta' = \int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\sigma(\theta',k)}e^{in\theta'}d\theta'.$$

According to Theorems 2.1 and 2.5, we get for every $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\begin{aligned}e^{in\theta}\tau(\theta,n) &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta'}\tau(\theta',n)e^{in\theta'}d\theta' \\ &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\sigma(\theta',k)}e^{in\theta'}d\theta'.\end{aligned}$$

Conversely, suppose that for every $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\tau(\theta,n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta'}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\sigma(\theta',k+n)e^{in\theta'}}d\theta'.$$

Then for every $m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta} e^{in\theta} \tau(\theta, n) d\theta = \int_{-\pi}^{\pi} e^{-im\theta'} \overline{\sigma(\theta', m)} e^{in\theta'} d\theta'$$

and by Theorem 3.1, the proof is complete. □

As applications of Theorems 3.1 and 3.2, we give necessary and sufficient conditions on the symbols of bounded linear operators $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to ensure that T is self-adjoint.

Corollary 3.3 *A bounded linear operator $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is self-adjoint if and only if for all $n, m \in \mathbb{Z}$,*

$$\int_{-\pi}^{\pi} e^{-im\theta} \sigma(\theta, n) e^{in\theta} d\theta = \int_{-\pi}^{\pi} e^{-im\theta} \overline{\sigma(\theta, m)} e^{in\theta} d\theta,$$

where σ is the symbol of T.

Corollary 3.4 *If $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator such that for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$, its corresponding symbol $\sigma(\theta, n) = \sigma(n) \in \mathbb{R}$ or $\sigma(\theta, n) = \sigma(\theta) \in \mathbb{R}$, then T is self-adjoint.*

In the following theorem, we show that the converse of Corollary 3.4 is true as well.

Theorem 3.5 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator such that the corresponding symbol σ is a real-valued measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then T is self-adjoint if and only if σ depends only on n or θ.*

Proof Suppose that σ is a real-valued measurable function such that σ depends only on n or θ. Then using Corollary 3.4, T is self-adjoint. Conversely, let σ be a real-valued measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is self-adjoint. Then according to Corollary 3.3, by changing variables, we have for all $m, k \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-ik\theta} \sigma(\theta, m)\, d\theta = \int_{-\pi}^{\pi} e^{-ik\theta} \sigma(\theta, m+k)\, d\theta. \tag{3.2}$$

Replacing m by $m + k$ in (3.2), we obtain

$$\int_{-\pi}^{\pi} e^{-ik\theta} \sigma(\theta, m+k) d\theta = \int_{-\pi}^{\pi} e^{-ik\theta} \sigma(\theta, m+2k)\, d\theta, \quad k, m \in \mathbb{Z}.$$

Repeating the process, we get

$$\int_{-\pi}^{\pi} e^{-ik\theta}\sigma(\theta,m)d\theta = \int_{-\pi}^{\pi} e^{-ik\theta}\sigma(\theta,m+nk)d\theta, \quad k,n,m\in\mathbb{Z}, \tag{3.3}$$

which means that the Fourier transform of σ does not depend on the second variable. Therefore σ depends only on n or θ. □

For bounded linear operators from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$, we have the following theorem, which follows from Theorem 3.1 and the proof of Theorem 3.5.

Theorem 3.6 *Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1, p_2 < \infty$, be a bounded linear operator with symbol σ. Then σ depends only on n or θ if and only if for all $(\theta,n)\in\mathbb{S}^1\times\mathbb{Z}$,*

$$\overline{\sigma^*(\theta,n)} = \overline{\sigma}^*(\theta,n),$$

where σ^* is the symbol of T^* and $\overline{\sigma}^*$ is the symbol of $T^*_{\overline{\sigma}}$.

Now, we investigate a bounded linear operator T with symbol σ given by

$$\sigma(\theta,n) = \sigma_1(n)\sigma_2(\theta), \quad (\theta,n)\in\mathbb{S}^1\times\mathbb{Z},$$

where σ_1 and σ_2 are measurable functions on, respectively, $\mathbb{Z}$ and $\mathbb{S}^1$. The following corollary follows from Corollary 3.3 immediately.

Corollary 3.7 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator such that the measurable function σ on $\mathbb{S}^1\times\mathbb{Z}$ given by*

$$\sigma(\theta,n) = \sigma_1(n)\sigma_2(\theta), \quad (\theta,n)\in\mathbb{S}^1\times\mathbb{Z},$$

is its symbol. Then T is self-adjoint if and only if for all $n,m\in\mathbb{Z}$,

$$\sigma_1(n)\int_{-\pi}^{\pi} e^{-im\theta}\sigma_2(\theta)e^{in\theta}d\theta = \overline{\sigma_1(m)}\int_{-\pi}^{\pi} e^{-im\theta}\overline{\sigma_2(\theta)}e^{in\theta}d\theta.$$

Theorem 3.8 *Let σ_1 be a measurable function on $\mathbb{Z}$ and σ_2 be a measurable function on $\mathbb{S}^1$. Let σ be the measurable function on $\mathbb{S}^1\times\mathbb{Z}$ defined by*

$$\sigma(\theta,n) = \sigma_1(n)\sigma_2(\theta), \quad (\theta,n)\in\mathbb{S}^1\times\mathbb{Z}.$$

Suppose that σ is the symbol of a bounded linear operator $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ and σ_2 is a nonconstant real-valued function. Then T is self-adjoint if and only if σ_1 is a real constant.

Proof Suppose that $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a self-adjoint operator. Then by Corollary 3.3, we get for all $n, m \in \mathbb{Z}$,

$$\begin{aligned}\sigma_1(n)\int_{-\pi}^{\pi} e^{-im\theta}\sigma_2(\theta)e^{in\theta}d\theta &= \overline{\sigma_1(m)}\int_{-\pi}^{\pi} e^{-im\theta}\overline{\sigma_2(\theta)}e^{in\theta}d\theta \\ &= \overline{\sigma_1(m)}\int_{-\pi}^{\pi} e^{-im\theta}\sigma_2(\theta)e^{in\theta}d\theta.\end{aligned}$$

Then for all $n, m \in \mathbb{Z}$, we get

$$\sigma_1(n) = \overline{\sigma_1(m)}.$$

This means that σ_1 is a real constant. Conversely, if σ_1 is a real constant, then by Theorem 3.5, T is self-adjoint. □

In the preceding theorem, the condition that σ_2 is a real-valued function cannot be removed. Indeed, let σ_2 be a complex-valued function on $\mathbb{S}^1$. Then it is easy to see that $T_{\sigma_2} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ cannot be self-adjoint.

In order to give another formula for the symbol of the adjoint of a bounded linear operator, we need the following theorem.

Theorem 3.9 *Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1,\ p_2 < \infty$, be a bounded linear operator with symbol σ. Then for every $f \in L^{p_1}(\mathbb{S}^1)$ and $\theta \in [-\pi, \pi]$,*

$$(Tf)(\theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)\hat{f}(k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\tau(\theta', k)}f(\theta')d\theta',$$

where τ is the symbol of T^* and is given by the formula given in Theorem 3.2.

Proof For every $f \in L^{p_1}(\mathbb{S}^1)$ and $\theta \in [-\pi, \pi]$, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\tau(\theta', k)}f(\theta')d\theta' &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} \overline{(T^*e_k)(\theta')}f(\theta')d\theta' \\ &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} e^{-ik\theta'}(Tf)(\theta')d\theta' \\ &= 2\pi(Tf)(\theta) \\ &= 2\pi\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)\hat{f}(k).\end{aligned}$$

□

Theorem 3.10 *Let $T : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$, $1 < p_1,\ p_2 < \infty$, be a bounded linear operator with symbol σ. Let τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then τ is the symbol of T^* if and only if*

$$\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)e^{-ik\theta'} = \sum_{k=-\infty}^{\infty} e^{ik\theta}\overline{\tau(\theta', k)}e^{-ik\theta'}, \quad \theta, \theta' \in [-\pi, \pi].$$

Proof Suppose that τ is the symbol of T^*. Then using Theorem 3.9, we get for every $f \in L^{p_1}(\mathbb{S}^1)$ and $\theta \in [-\pi, \pi]$,

$$(Tf)(\theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)\hat{f}(k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\tau(\theta', k)}f(\theta')d\theta'.$$

In fact, for all $f \in L^{p_1}(\mathbb{S}^1)$ and $\theta \in [-\pi, \pi]$,

$$\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)e^{-ik\theta'}f(\theta')d\theta' = \int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta}\overline{\tau(\theta', k)}e^{-ik\theta'}f(\theta')d\theta'.$$

So,

$$\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)e^{-ik\theta'} = \sum_{k=-\infty}^{\infty} e^{ik\theta}\overline{\tau(\theta', k)}e^{-ik\theta'}, \quad \theta, \theta' \in [-\pi, \pi].$$

Conversely, suppose that

$$\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)e^{-ik\theta'} = \sum_{k=-\infty}^{\infty} e^{ik\theta}\overline{\tau(\theta', k)}e^{-ik\theta'}, \quad \theta, \theta' \in [-\pi, \pi].$$

Then for every $f \in L^{p_1}(\mathbb{S}^1)$,

$$\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)e^{-ik\theta'}f(\theta') = \sum_{k=-\infty}^{\infty} e^{ik\theta}\overline{\tau(\theta', k)}e^{-ik\theta'}f(\theta'), \quad \theta, \theta' \in [-\pi, \pi].$$

Integrating with respect to θ', we get

$$\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)\hat{f}(k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ik\theta}\int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\tau(\theta', k)}f(\theta')d\theta'$$

and the proof is complete. □

Corollary 3.11 *A bounded linear operator $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ with symbol σ is self-adjoint if and only if*

$$\sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k)e^{-ik\theta'} = \sum_{k=-\infty}^{\infty} e^{ik\theta}\overline{\sigma(\theta', k)}e^{-ik\theta'}, \quad \theta, \theta' \in [-\pi, \pi].$$

4 Eigenvalues, Eigenfunctions and Compact Operators

In this section, we give necessary and sufficient conditions to guarantee that bounded linear operators T from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$ for $1 < p < \infty$ have eigenvalues and eigenfunctions. Also, necessary and sufficient conditions are given for bounded linear operators from $L^2(\mathbb{S}^1)$ to $L^2(\mathbb{S}^1)$ to be compact or compact self-adjiont. As applications, we give formulas for the symbols of compact and compact self-adjoint operators.

In the following theorem, a necessary and sufficient condition is given for bounded linear operators $T : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1),\ 1 < p < \infty$, to have eigenvalues and eigenfunctions.

Theorem 4.1 *Let $T : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1),\ 1 < p < \infty$, be a bounded linear operator such that the measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ is its symbol. Then $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if there is a nonzero function $f \in L^p(\mathbb{S}^1)$ such that*

$$\int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)} f(\theta) d\theta = 2\pi\lambda \hat{f}(m), \quad m \in \mathbb{Z},$$

where τ is the symbol of T^*.

Proof Suppose that $T : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1),\ 1 < p < \infty$, is a bounded linear operator and $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then there is a nonzero function $f \in L^p(\mathbb{S}^1)$ such that

$$(Tf)(\theta) = \lambda f(\theta), \quad \theta \in [-\pi, \pi].$$

So, for every $m \in \mathbb{Z}$,

$$\begin{aligned} \int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)} f(\theta) d\theta &= \int_{-\pi}^{\pi} \overline{(T^* e_m)(\theta)} f(\theta) d\theta \\ &= \int_{-\pi}^{\pi} e^{-im\theta} (Tf)(\theta)\, d\theta \\ &= 2\pi\lambda \hat{f}(m). \end{aligned}$$

Conversely, suppose that there is a nonzero function $f \in L^p(\mathbb{S}^1)$ such that

$$\int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)} f(\theta) d\theta = 2\pi\lambda \hat{f}(m), \quad m \in \mathbb{Z}.$$

Then for every $m \in \mathbb{Z}$,

$$\begin{aligned} 2\pi\lambda\hat{f}(m) &= \int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)}f(\theta)d\theta \\ &= \int_{-\pi}^{\pi} \overline{(T^*e_m)(\theta)}f(\theta)d\theta \\ &= \int_{-\pi}^{\pi} e^{-im\theta}(Tf)(\theta)d\theta \\ &= 2\pi\widehat{(Tf)}(m). \end{aligned}$$

So, $Tf = \lambda f$ and the proof is complete □

We have the following immediate corollary.

Corollary 4.2 *Let $T : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$, $1 < p < \infty$, be a bounded linear operator with symbol σ. Then $f = e_n$, $n \in \mathbb{Z}$, is an eigenfunction of T if and only if for all $m \in \mathbb{Z}$,*

$$\int_{-\pi}^{\pi} e^{-im\theta}\overline{\tau(\theta, m)}e^{in\theta}d\theta = \begin{cases} \int_{-\pi}^{\pi} \overline{\tau(\theta, n)}d\theta, & n = m, \\ 0, & n \neq m, \end{cases}$$

where τ is the symbol of T^*. In this case,

$$\lambda_n = \int_{-\pi}^{\pi} \overline{\tau(\theta, n)}d\theta$$

is the eigenvalue corresponding to the eigenfunction e_n.

Using Theorem 3.5, we have another corollary of Theorem 4.1.

Theorem 4.3 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator such that the real-valued measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ is its symbol. Then T is self-adjoint if and only if for all $n \in \mathbb{Z}$, the function e_n is an eigenfunction of T corresponding to the eigenvalue $\sigma(n)$ or σ is a function of θ only.*

In the next theorem, we show that a nonzero function $\varphi \in L^2(\mathbb{S}^1)$ is an eigenfunction of a bounded linear operator $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ with symbol σ if and only if $\hat{\varphi}$ is an eigenfunction of a bounded linear operator $P : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ with symbol ρ, where ρ can be calculated in terms of σ.

Theorem 4.4 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator with symbol σ. Then a nonzero function $\varphi \in L^2(\mathbb{S}^1)$ is an eigenfunction of T if and only if $\hat{\varphi}$ is an eigenfunction of $P : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ with symbol ρ such that*

$$\rho(n,\theta) = \sum_{k=-\infty}^{\infty} e^{-ik\theta'} \int_{-\pi}^{\pi} e^{ik\theta'} \sigma(\theta', k+n)d\theta', \quad (n,\theta) \in \mathbb{Z} \times \mathbb{S}^1.$$

In this case, the corresponding eigenvalues of φ and $\hat{\varphi}$ are the same.

Proof Suppose that φ is an eigenfunction of $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ corresponding to the eigenvalue λ. Then by Theorem 4.1,

$$\begin{aligned} 2\pi\lambda\hat{\varphi}(n) &= \int_{-\pi}^{\pi} e^{-in\theta}\overline{\tau(\theta,n)}\varphi(\theta)d\theta \\ &= \int_{-\pi}^{\pi} e^{-in\theta}\overline{\tau(\theta,n)}(\mathcal{F}_{\mathbb{Z}}\hat{\varphi})(\theta)d\theta = (P\hat{\varphi})(n) \end{aligned}$$

for all $n \in \mathbb{Z}$, where $\rho : \mathbb{Z} \times \mathbb{S}^1 \to \mathbb{C}$ given by

$$\rho(n,\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\theta} \int_{-\pi}^{\pi} e^{ik\theta'} \sigma(\theta', k+n)d\theta', \quad (n,\theta) \in \mathbb{Z} \times \mathbb{S}^1,$$

is the symbol of the pseudo-differential operator $P : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$. Conversely, let $\hat{\varphi}$ be an eigenfunction of $P : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ with corresponding eigenvalue λ such that the function ρ on $\mathbb{Z} \times \mathbb{S}^1$ defined by

$$\rho(n,\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\theta} \int_{-\pi}^{\pi} e^{ik\theta'} \sigma(\theta', k+n)d\theta', \quad (n,\theta) \in \mathbb{Z} \times \mathbb{S}^1,$$

is the symbol of P. Then

$$\begin{aligned} 2\pi\lambda\hat{\varphi}(n) = (P\hat{\varphi})(n) &= \int_{-\pi}^{\pi} e^{-in\theta}\rho(n,\theta)(\mathcal{F}_{\mathbb{Z}}\hat{\varphi})(\theta)d\theta \\ &= \int_{-\pi}^{\pi} e^{-in\theta}\overline{\tau(\theta,n)}\varphi(\theta)d\theta \end{aligned}$$

for all $n \in \mathbb{Z}$, where by Theorem 3.2,

$$\tau(\theta,n) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik\theta} \int_{-\pi}^{\pi} e^{-ik\theta'} \overline{\sigma(\theta', k+n)}d\theta', \quad (\theta,n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

By Theorem 3.2, τ is the symbol of the adjoint of $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ with symbol σ. Using Theorem 4.1, the proof is complete. $\square$

The following theorem tells us which compact operators $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ are self-adjoint.

Theorem 4.5 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then T is self-adjoint if and only if we can find an orthonormal set $\{\varphi_k\}_{k=1}^\infty$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=1}^\infty$ of real numbers such that*

$$\lim_{k\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=1}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Proof Suppose that $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a self-adjoint operator. Then we can find an orthonormal set $\{\varphi_k\}_{k=1}^\infty$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=1}^\infty$ of real numbers such that

$$\lim_{k\to\infty} \lambda_k = 0$$

and for every $f \in L^2(\mathbb{S}^1)$,

$$(Tf)(\theta) = \sum_{k=1}^{\infty} e^{ik\theta}\sigma(\theta, k)\hat{f}(k) = \sum_{k=1}^{\infty} \varphi_k(\theta)\lambda_k \int_{-\pi}^{\pi} \overline{\varphi_k(\theta')}f(\theta')d\theta'$$

for all $\theta \in [-\pi, \pi]$. Now, let $f = e_n$, $n \in \mathbb{Z}$. Then

$$e^{in\theta}\sigma(\theta, n) = 2\pi \sum_{k=1}^{\infty} \varphi_k(\theta)\lambda_k\overline{\widehat{\varphi_k}(n)}, \quad \theta \in [-\pi, \pi].$$

So, for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=1}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Conversely, suppose that we can find an orthonormal set $\{\varphi_k\}_{k=1}^\infty$ in $L^2(\mathbb{S}^1)$ and a sequence of real numbers $\{\lambda_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \lambda_k = 0$$

and for every $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=1}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Then by Corollary 3.3, T is self-adjoint. □

We have the following corollaries of the above theorem.

Corollary 4.6 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then T is self-adjoint if and only if we can find an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence of real numbers $\{\lambda_k\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \lambda_k = 0$$

and for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\sigma(\theta, n) = 2\pi \sum_{k=1}^{\infty} e^{-in\theta}\varphi_k(\theta)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Corollary 4.7 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. If T is a self-adjoint operator, then we can find an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\varphi_k\}_{k=1}^{\infty}$ of real numbers such that*

$$\lim_{k\to\infty} \lambda_k = 0$$

and for all $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \sigma(\theta, n)d\theta = 4\pi^2 \sum_{k=1}^{\infty} \lambda_k|\widehat{\varphi}_k(n)|^2.$$

Corollary 4.8 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. If T is self-adjoint, then*

$$\frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \sigma(\theta, n)d\theta = \sum_{k=1}^{\infty} \lambda_k,$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of T.

Corollary 4.9 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Suppose that T is self-adjoint. Then we can find an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of real numbers such that*

$$\lim_{k\to\infty} \lambda_k = 0$$

and for all $n \in \mathbb{Z}$,

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta = \sum_{k=1}^{\infty} \lambda_k^2 \; |\widehat{\varphi_k}(n)|^2$$

and hence

$$\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta = \sum_{k=-\infty}^{\infty} \lambda_k^2 = \|T\|_{HS}^2,$$

where φ_k is an eigenfunction corresponding to the eigenvalue λ_k of T for all $k \in \mathbb{N}$ and $\|T\|_{HS}$ is the Hilbert–Schmidt norm of T.

In the following theorems, we give other necessary and sufficient conditions on the symbols σ to ensure that the corresponding compact operators from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ are self-adjoint.

Theorem 4.10 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then T is self-adjoint if and only if we can find an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of real numbers such that*

$$\lim_{k\to\infty} \lambda_k = 0$$

and for all $\theta, \theta' \in [-\pi, \pi]$,

$$\sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(\theta, n)e^{-in\theta'} = \sum_{k=-\infty}^{\infty} \varphi_k(\theta)\lambda_k\overline{\varphi_k}(\theta').$$

Theorem 4.10 follows from Theorem 4.5 and Corollary 4.6.

Theorem 4.11 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then T is self-adjoint if and only if we can find an orthonormal set $\{\varphi_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of real numbers such that*

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$*,*

$$\int_{-\pi}^{\pi} e^{-im\theta} \overline{\sigma(\theta, m)} \varphi_n(\theta) d\theta = (2\pi) \lambda_n \widehat{\varphi_n}(m).$$

Proof Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be self-adjoint. Then we can find an orthonormal set $\{\varphi_k\}_{k \in \mathbb{Z}}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda\}_{k=-\infty}^{\infty}$ of real numbers such that

$$\lim_{|k| \to \infty} \lambda_k = 0$$

and for every $m \in \mathbb{Z}$,

$$e^{im\theta} \sigma(\theta, m) = 2\pi \sum_{k=-\infty}^{\infty} \varphi_k(\theta) \lambda_k \overline{\widehat{\varphi_k}(m)}.$$

So, for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta} \overline{\sigma(\theta, m)} \varphi_n(\theta) d\theta = 2\pi \lambda_n \widehat{\varphi_n}(m).$$

Conversely, suppose that we can find an orthonormal set $\{\varphi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of real numbers such that

$$\lim_{|k| \to \infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta} \overline{\sigma(\theta, m)} \varphi_n(\theta) d\theta = 2\pi \lambda_n \widehat{\varphi_n}(m).$$

Let $k \in \mathbb{Z}$. Then we have

$$\overline{\widehat{\varphi_n}(k)} \int_{-\pi}^{\pi} e^{-im\theta} \overline{\sigma(\theta, m)} \varphi_n(\theta) d\theta = 2\pi \lambda_n \widehat{\varphi_n}(m) \overline{\widehat{\varphi_n}(k)}$$

and

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-im\theta} \overline{\sigma(\theta, m)} \varphi_n(\theta) \overline{\widehat{\varphi_n}(k)} d\theta = 2\pi \sum_{n=-\infty}^{\infty} \widehat{\varphi_n}(m) \lambda_n \overline{\widehat{\varphi_n}(k)}.$$

Since for every $k \in \mathbb{Z}$,

$$e^{ik\theta} = 2\pi \sum_{n=-\infty}^{\infty} \varphi_n(\theta) \overline{\widehat{\varphi_n}(k)}, \quad \theta \in [-\pi, \pi],$$

it follows from Theorem 4.5 that T is self-adjoint. $\square$

Theorem 4.12 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then we can find an orthonormal set $\{\varphi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of complex numbers such that*

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\overline{\sigma(\theta,m)}\sigma(\theta,n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Proof Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator. Since T^*T is self-adjoint, we can find an orthonormal set $\{\varphi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of real numbers such that

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}e^{in\theta}\rho(\theta,n)d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)},$$

where ρ is the symbol of T^*T. But

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-im\theta}e^{in\theta}\rho(\theta,n)d\theta &= \int_{-\pi}^{\pi} e^{-im\theta}(T^*Te_n)(\theta)d\theta \\ &= \int_{-\pi}^{\pi} e^{-im\theta}\overline{\sigma(\theta,m)}\sigma(\theta,n)e^{in\theta}d\theta.\end{aligned}$$

Therefore

$$\int_{-\pi}^{\pi} e^{-im\theta}\overline{\sigma(\theta,m)}\sigma(\theta,n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

□

Corollary 4.13 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then*

$$\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} |\sigma(\theta,n)|^2 d\theta = \sum_{k=-\infty}^{\infty} \lambda_k,$$

where λ_k, $k \in \mathbb{Z}$, are the eigenvalues of T^*T.

Now, we give necessary and sufficient conditions for bounded linear operators $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to be compact. To do this, we need the following well-known theorem.

Theorem 4.14 *Let $\{\sigma_k\}_{k=-\infty}^{\infty}$ and $\{e_k\}_{k=-\infty}^{\infty}$ be orthonormal sets in the Hilbert space H, and let $\{\lambda_k\}_{k=-\infty}^{\infty}$ be a sequence of complex numbers such that*

$$\lim_{|k|\to\infty} \lambda_k = 0.$$

Let $T : H \to H$ be defined by

$$Tx = \sum_{k=-\infty}^{\infty} \lambda_k \langle x, e_k \rangle \sigma_k \quad x \in H.$$

Then T is compact.

Theorem 4.15 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator with symbol σ. Then T is a compact operator if and only if there exist orthonormal sets $\{\varphi_k\}_{k=-\infty}^{\infty}$ and $\{\psi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence of positive numbers $\{\lambda_k\}_{k=-\infty}^{\infty}$ such that*

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta} \sigma(\theta, n) e^{in\theta} d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\psi_k}(m) \lambda_k \overline{\widehat{\varphi_k}(n)}.$$

Proof Suppose that $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a compact operator. Then we can find a partial isometry V and a positive operator $|T| = (T^*T)^{1/2}$ such that $T = V|T|$. Since $|T|$ is self-adjoint, we can find an orthonormal basis $\{\varphi_k\}_{k=-\infty}^{\infty}$ for $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of positive numbers such that for all $n, m \in \mathbb{Z}$,

$$(|T|e_n)(\theta) = 2\pi \sum_{k=-\infty}^{\infty} \varphi_k(\theta) \lambda_k \overline{\widehat{\varphi_k}(n)}, \quad \theta \in [-\pi, \pi].$$

Then

$$e^{in\theta} \sigma(\theta, n) = (Te_n)(\theta) = (V|T|e_n)(\theta) = 2\pi \sum_{k=-\infty}^{\infty} \psi_k(\theta) \lambda_k \overline{\widehat{\varphi_k}(n)}$$

for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$, where $\{\psi_k\}_{k=-\infty}^{\infty} = \{V\varphi_k\}_{k=-\infty}^{\infty}$ is an orthonormal set in $L^2(\mathbb{S}^1)$. So,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\psi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}$$

for all $n, m \in \mathbb{Z}$. Conversely, suppose that there are orthonormal sets $\{\varphi_k\}_{k=-\infty}^{\infty}$ and $\{\psi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of positive numbers with

$$\lim_{|k|\to\infty} \lambda_k = 0$$

such that for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\psi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Then for every $f \in L^2(\mathbb{S}^1)$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-im\theta'}(Tf)(\theta')d\theta' &= \sum_{n=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} e^{-im\theta'}\sigma(\theta', n)e^{in\theta'}d\theta'\right)\hat{f}(n) \\ &= \sum_{k=-\infty}^{\infty} \widehat{\psi_k}(m)\lambda_k \int_{-\pi}^{\pi} \overline{\varphi_k(\theta')}f(\theta')d\theta'\end{aligned}$$

for all $m \in \mathbb{Z}$. So,

$$(Tf)(\theta) = \sum_{k=-\infty}^{\infty} \psi_k(\theta)\lambda_k \int_{-\pi}^{\pi} \overline{\varphi_k(\theta')}f(\theta')d\theta', \quad \theta \in [-\pi, \pi].$$

Therefore T is compact by Theorem 4.14. □

By taking $\varphi_n = e_n$ in the above theorem, we have the following example.

Example 4.16 Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator with symbol σ. If $\{\psi_k\}_{k=-\infty}^{\infty}$ is an orthonormal set in $L^2(\mathbb{S}^1)$ and $\{\lambda_k\}_{k=-\infty}^{\infty}$ is a sequence of complex numbers such that

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\sigma(\theta, n) = e^{-in\theta}\psi_n(\theta)\lambda_n,$$

then T is a compact operator.

Now, using Theorem 4.15, we are able to give a formula for the symbols of compact operators.

Corollary 4.17 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator with symbol σ. Then T is a compact operator if and only if there exist orthonormal sets $\{\varphi_k\}_{k=-\infty}^{\infty}$ and $\{\psi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of complex numbers such that*

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\sigma(\theta, n) = 2\pi \sum_{k=-\infty}^{\infty} e^{-in\theta} \psi_k(\theta) \lambda_k \overline{\widehat{\varphi_k}(n)}.$$

Corollary 4.18 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator with symbol σ. Then there are orthonormal sets $\{\varphi_k\}_{k=-\infty}^{\infty}$ and $\{\psi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of complex numbers such that*

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \sigma(\theta, n) d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\psi_k}(n) \lambda_k \overline{\widehat{\varphi_k}(n)}.$$

5 Factorizations of Compact Operators

We begin with the following remark.

Remark 5.1 If $P : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ and $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ are two bounded linear operators such that ρ is the symbol of TP, then for all $(\theta, n) \in \mathbb{S} \times \mathbb{Z}$,

$$\begin{aligned} e^{in\theta}\rho(\theta, n) &= (TPe_n)(\theta) \\ &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k) \int_{-\pi}^{\pi} e^{-ik\theta'}(Pe_n)(\theta')d\theta' \\ &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta, k) \int_{-\pi}^{\pi} e^{-ik\theta'}\overline{\eta(\theta', k)}e^{in\theta'}d\theta', \end{aligned}$$

where σ is the symbol T and η is the symbol of P^*. We use this fact to give necessary and sufficient conditions for bounded linear operators to be compact and to show that every compact operator from $L^2(\mathbb{S}^1)$ to $L^2(\mathbb{S}^1)$ can be written as the product of two compact operators.

Theorem 5.2 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator with symbol σ. Then T is a compact operator if and only if there exist two compact operators S and P such that*

$$T = PS^*.$$

Proof Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a compact operator. Then there exist orthonormal sets $\{\varphi_k\}_{k=-\infty}^{\infty}$ and $\{\psi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of positive numbers such that

$$\lim_{|k|\to\infty|} \lambda_k = 0$$

and for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$e^{in\theta}\sigma(\theta, n) = 2\pi \sum_{k=-\infty}^{\infty} \psi_k(\theta)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Now, let ω and η be measurable functions on $\mathbb{S}^1 \times \mathbb{Z}$ defined by

$$\omega(\theta, k) = e^{-ik\theta}\psi_k(\theta)\lambda_k^{1/2}$$

and

$$\eta(\theta, k) = e^{-ik\theta}\varphi_k(\theta)\lambda_k^{1/2}$$

for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$. Then

$$e^{in\theta}\sigma(\theta, n) = \sum_{k=-\infty}^{\infty} e^{ik\theta}\omega(\theta, k) \int_{-\pi}^{\pi} e^{-ik\theta'}\ \overline{\eta(\theta', k)}e^{in\theta'}d\theta'$$

for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$. According to Remark 5.1, σ is the symbol of the product of two operators P and S^*, where ω is the symbol of P and η is the symbol of S. According to Example 4.16, both corresponding operators are compact. The converse is obvious. □

In the next theorem, we give necessary and sufficient conditions for bounded linear operators $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to be compact self-adjoint.

Theorem 5.3 *Let $T : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be a bounded linear operator with symbol σ. Then T is a compact self-adjoint operator if and only if we can find an orthonormal set $\{\varphi_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of positive numbers such that*

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\varphi}_k(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Proof Suppose that we can find an orthonormal basis $\{\varphi_k\}_{k=-\infty}^{\infty}$ for $L^2(\mathbb{S}^1)$ and a sequence $\{\lambda_k\}_{k=-\infty}^{\infty}$ of positive numbers such that

$$\lim_{|k|\to\infty} \lambda_k = 0$$

and for all $n, m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta, n)e^{in\theta}d\theta = 4\pi^2 \sum_{k=-\infty}^{\infty} \widehat{\varphi_k}(m)\lambda_k\overline{\widehat{\varphi_k}(n)}.$$

Then by Theorem 4.15 and Corollary 3.3, T is a a compact self-adjoint operator. The converse follows from Theorem 4.5 ☐

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