The Green Function of the Sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^1$

Let g be the Green function of $\mathcal{L}$ on $\mathbb{H}^1$. Then for all f in $L^2(\mathbb{H}^1)$, $\mathcal{L}^{-1}f = f *_{\mathbb{H}^1} g$. $\therefore$

$$(\mathcal{L}^{-1}f)^\tau = (2\pi)^{1/2}(f^\tau *_{\tau/4} g^\tau) = (2\pi)^{1/2}(g^\tau *_{-\tau/4} f^\tau). \therefore$$

$$(L_\tau^{-1} f^\tau)(z) = (2\pi)^{1/2}\int_{\mathbb{C}} g^\tau(z-w)\, e^{-i\frac{\tau}{4}[z,w]} f^\tau(w)\,dw$$

$$= \int_{\mathbb{C}} G_\tau(z,w) f^\tau(w)\,dw, \quad z\in\mathbb{C}.$$

$$\therefore g^\tau(z-w) = (2\pi)^{-1/2}\frac{1}{4\pi}K_0\left(\frac{1}{4}|\tau||z-w|^2\right), \quad z,w\in\mathbb{C}.$$

$$\therefore g^\tau(z) = (2\pi)^{-1/2}\frac{1}{4\pi}K_0\left(\frac{1}{4}|\tau||z|^2\right), \quad z\in\mathbb{C}.$$

$\therefore$

$$g(z,t)$$
$$= \frac{1}{8\pi^2}\int_{-\infty}^{\infty} e^{-it\tau} K_0\left(\frac{1}{4}|\tau||z|^2\right)d\tau$$
$$= \frac{1}{8\pi^2}\int_{-\infty}^{\infty} e^{-it\tau}\left(\int_0^\infty e^{-\frac{1}{4}|\tau||z|^2\cosh\delta}\,d\delta\right)d\tau$$
$$= \frac{1}{8\pi^2}\int_0^\infty \Bigg(\underbrace{\int_{-\infty}^{\infty} e^{-it\tau}\, e^{-\frac{1}{4}|\tau||z|^2\cosh\delta}\,d\tau}_{I}\Bigg)d\delta.$$

But

$$\int_{-\infty}^{0} e^{-it\tau} e^{\frac{1}{4}\tau|z|^2\cosh\delta}\, d\tau$$

$$= \frac{e^{\tau\left(\frac{1}{4}|z|^2\cosh\delta - it\right)}}{\frac{1}{4}|z|^2\cosh\delta - it}\Bigg|_{-\infty}^{0} = \frac{1}{\frac{1}{4}|z|^2\cosh\delta - it}$$

and

$$\int_{0}^{\infty} e^{-it\tau} e^{-\frac{1}{4}\tau|z|^2\cosh\delta}\, d\tau$$

$$= -\frac{e^{-\tau\left(\frac{1}{4}|z|^2\cosh\delta + it\right)}}{\frac{1}{4}|z|^2\cosh\delta + it}\Bigg|_{0}^{\infty} = \frac{1}{\frac{1}{4}|z|^2\cosh\delta + it}$$

So,

$$I = \frac{\frac{1}{2}|z|^2\cosh\delta}{(|z|^4/16)\cosh^2\delta + t^2}.$$

$$\therefore\quad g(z,t) = \frac{1}{8\pi^2}\int_{0}^{\infty} \frac{\frac{1}{2}|z|^2\cosh\delta}{(|z|^4/16)\cosh^2\delta + t^2}\, d\delta$$

$$= \frac{|z|^2}{16\pi^2}\int_{0}^{\infty} \frac{\cosh\delta}{(|z|^4/16)\cosh^2\delta + t^2}\, d\delta$$

$$= \frac{1}{\pi^2|z|^2}\int_{0}^{\infty} \frac{\cosh\delta}{\cosh^2\delta + (16t^2/|z|^4)}\, d\delta.$$

Let $\phi = \sinh \delta$. Then $\cosh^2 - \sinh^2 = 1$ and $d\phi = \cosh\delta\, d\delta$.

So

$$g(z,t) = \frac{1}{\pi^2 |z|^2} \int_0^\infty \frac{1}{\phi^2 + 1 + (16t^2/|z|^4)}\, d\phi$$

$$= \frac{1}{2\pi^2 |z|^2} \int_{-\infty}^\infty \frac{1}{\phi^2 + 1 + (16t^2/|z|^4)}\, d\phi$$

$$= \frac{1}{2\pi^2 |z|^2} \frac{1}{\sqrt{1 + (16t^2/|z|^4)}} \tan^{-1} \frac{\phi}{\sqrt{1+(16t^2/|z|^4)}} \Bigg]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi^2 |z|^2} \frac{1}{\sqrt{1+(16t^2/|z|^4)}}\, \pi$$

$$= \frac{1}{2\pi \sqrt{|z|^4 + 16t^2}} = \frac{1}{2\pi} \frac{1}{\sqrt{|z|^4 + 16t^2}}.$$

Remarks: The Newtonian potential of $-\Delta$ on $\mathbb{R}^3$ is

$$\frac{1}{4\pi} \frac{1}{\sqrt{|z|^2 + t^2}},$$

which is $\frac{1}{4\pi}$ times the inverse of the distance of (z,t) from the origin. By analogy, the distance of (z,t) from the origin in $\mathbb{H}^1$ is $\sqrt{|z|^4 + 16t^2}$.