Lecture 23

The Green Function of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$

Let $\tau \in \mathbb{R}\setminus\{0\}$. Let G_τ be the Green function of L_τ. Then for all f in $L^2(\mathbb{C})$,

$$(L_\tau^{-1} f)(z) = \int_{\mathbb{C}} G_\tau(z,w) f(w)\,dw, \quad z \in \mathbb{C}.$$

To find an explicit formula for G_τ, we recall that for all f in $L^2(\mathbb{C})$,

$$(e^{-\rho L_\tau} f)(z) = \int_{\mathbb{C}} \kappa_\rho^\tau(z,w) f(w)\,dw, \quad z \in \mathbb{C}.$$

So,

$$(L_\tau^{-1} f)(z) = \left(\left(\int_0^\infty e^{-\rho L_\tau}\,d\rho\right) f\right)(z)$$

$$= \left(\int_0^\infty e^{-\rho L_\tau} f\,d\rho\right)(z) = \int_0^\infty (e^{-\rho L_\tau} f)(z)\,d\rho$$

$$= \int_0^\infty \left(\int_{\mathbb{C}} \kappa_\rho^\tau(z,w) f(w)\,dw\right) d\rho = \int_{\mathbb{C}} \left(\int_0^\infty \kappa_\rho^\tau(z,w)\,d\rho\right) f(w)\,dw.$$

$$\therefore\; G_\tau(z,w) = \int_0^\infty \kappa_\rho^\tau(z,w)\,d\rho, \quad z, w \in \mathbb{C}.$$

<u>Theorem</u>: Let $\tau \in \mathbb{R}\setminus\{0\}$. Then

$$G_\tau(z,w) = \frac{1}{4\pi} e^{-i\frac{\tau}{4}[z,w]} K_0\left(\frac{1}{4}|\tau||z-w|^2\right), \quad z,w \in \mathbb{C},$$

where K_0 is the modified Bessel function of order 0 given by

$$K_0(x) = \int_0^\infty e^{-x\cosh t}\,dt, \quad x>0.$$

<u>Proof</u>: Let $z,w \in \mathbb{C}$. Then

$$G_\tau(z,w) = \int_0^\infty \kappa_\rho^\tau(z,w)\,d\rho$$
$$= \int_0^\infty \frac{1}{4\pi}\frac{|\tau|}{\sinh(|\tau|\rho)} e^{-\frac{1}{4}|\tau||z-w|^2\coth(|\tau|\rho)} e^{-i\frac{\tau}{4}[z,w]}\,d\rho$$
$$= \frac{1}{4\pi} e^{-i\frac{\tau}{4}[z,w]} \int_0^\infty \frac{|\tau|}{\sinh(|\tau|\rho)} e^{-\frac{1}{4}|\tau||z-w|^2\coth(|\tau|\rho)}\,d\rho.$$

Let $v = \coth(|\tau|\rho)$. Then

$$dv = -\operatorname{csch}^2(|\tau|\rho)|\tau|\,d\rho = \frac{-|\tau|}{\sinh^2(|\tau|\rho)}\,d\rho.$$

Since

$$\coth^2 = 1 + \operatorname{csch}^2,$$

$$\therefore\ G_\tau(z,w) = \frac{1}{4\pi} e^{-i\frac{\tau}{4}[z,w]} \int_1^\infty \frac{1}{(v^2-1)^{1/2}} e^{-\frac{1}{4}|z-w|^2 v}\,dv.$$

By a formula in the book on formulas by Magnus, Oberhettinger and Soni,

$$\int_1^\infty (v^2-1)^{\gamma-1} e^{-\mu v}\, dv = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\gamma-1}\Gamma(\gamma)K_{\gamma-\frac{1}{2}}(\mu).$$

Let $\gamma = \frac{1}{2}$, $\mu = \frac{1}{4}|\tau||z-w|^2$. Then

$$G_\tau(z,w)$$

$$= \frac{1}{4\pi} e^{-i\frac{\tau}{4}[z,w]} \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)K_0\left(\frac{1}{4}|\tau||z-w|^2\right)$$

$$= \frac{1}{4\pi} e^{-i\frac{\tau}{4}[z,w]} K_0\left(\frac{1}{4}|\tau||z-w|^2\right), \quad z,w \in \mathbb{C}.$$

——— . ———