Lecture 22

The sub-Laplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^1$ is given by

$$\mathcal{L} = -\Delta - \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}.$$

The heat kernel K_ρ, $\rho>0$, of $\mathcal{L}$ is the function on $\mathbb{H}^1$ for which

$$e^{-\rho\mathcal{L}} f = f *_{\mathbb{H}^1} K_\rho, \quad f \in L^2(\mathbb{H}^1).$$

$$\therefore \; e^{-\rho L_\tau} f^\tau = (2\pi)^{1/2}\left(f^\tau *_{\tau/4} K_\rho^\tau\right)$$
$$= (2\pi)^{1/2}\left(K_\rho^\tau *_{-\tau/4} f^\tau\right).$$

But we have shown that

$$\left(e^{-\rho L_\tau} f^\tau\right)(z) = \int_{\mathbb{C}} K_\rho^\tau(z,w) f^\tau(w)\,dw, \quad f \in L^2(\mathbb{R}^2).$$
$$= (2\pi)^{1/2}\int_{\mathbb{C}} K_\rho^\tau(z-w)\, e^{-i\frac{\tau}{4}[z,w]} f^\tau(w)\,dw, \quad f\in L^2(\mathbb{R}^2).$$

$$\therefore \; K_\rho^\tau(z-w)\, e^{-i\frac{\tau}{4}[z,w]}$$
$$= (2\pi)^{-1/2}\frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)} e^{-\frac{\tau}{4}|z-w|^2\coth(\tau\rho)}\, e^{-i\frac{\tau}{4}[z,w]}, \quad z,w\in\mathbb{C}.$$

$$\therefore \; K_\rho^\tau(z-w) = (2\pi)^{-1/2}\frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)} e^{-\frac{\tau}{4}|z-w|^2\coth(\tau\rho)}, \quad z,w\in\mathbb{C}.$$

<u>Theorem</u>: Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for all $\rho > 0$,

$$K_\rho^\tau(z) = (2\pi)^{-1/2} \frac{1}{4\pi} \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{\tau}{4}|z|^2 \coth(\tau\rho)}, \quad z \in \mathbb{C}$$

<u>Theorem</u>: The heat kernel K_ρ of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^1$ is given by

$$K_\rho(z,t) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{-it\tau} \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{\tau}{4}|z|^2 \coth(\tau\rho)} \, d\tau.$$

The Green Function of $\mathcal{L}$ on $\mathbb{H}^1$

The Green function $\mathcal{G}$ of $\mathcal{L}$ on $\mathbb{H}^1$ is the function on $\mathbb{H}^1$ given by

$$\mathcal{L}^{-1} f = f *_{\mathbb{H}^1} \mathcal{G}.$$

Taking the inverse Fourier transform with respect to t, we get

$$L_\tau^{-1} f^\tau = (f^\tau *_{\tau/4} \mathcal{G}^\tau)(2\pi)^{1/2}$$
$$= (2\pi)^{1/2} (\mathcal{G}^\tau *_{-\tau/4} f^\tau)$$

So, we can find $\mathcal{G}$ if we can find the Green function of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$.