Lecture 21

Some Basic Functional Analysis: Let X be a complex and separable Hilbert space, e.g., $L^2(\mathbb{R}^n)$. Let $\{\varphi_1, \varphi_2, \dots\}$ be an orthonormal basis for X consisting of all eigenvectors of a closed linear operator A from X into X, e.g, the twisted Laplacian L_τ, $\tau \in \mathbb{R}\setminus\{0\}$. Let λ_k be an eigenvalue of A corresponding to φ_k for all $k \in \mathbb{N}$. Then for all $x \in X$,

$$x = \sum_{k=1}^{\infty} (x, \varphi_k)\varphi_k.$$

$$\therefore\ Ax = \sum_{k=1}^{\infty} (x, \varphi_k) A\varphi_k = \sum_{k=1}^{\infty} (x, \varphi_k)\lambda_k \varphi_k$$

$$= \sum_{k=1}^{\infty} \lambda_k (x, \varphi_k)\varphi_k.$$

Now, for all $\rho > 0$,

$$e^{-\rho A} x = \sum_{k=1}^{\infty} e^{-\rho\lambda_k}(x, \varphi_k)\varphi_k.$$

Also, if $\lambda_k \neq 0$ for all $k \in \mathbb{N}$,

$$A^{-1}x = \sum_{k=1}^{\infty} \frac{1}{\lambda_k}(x, \varphi_k)\varphi_k.$$

Recall: ① Let $\tau \in \mathbb{R}\setminus\{0\}$. Then

$$L_\tau e^\tau_{j,k} = (2k+1)|\tau| e^\tau_{j,k}, \quad j,k = 0,1,2,\cdots.$$

② $\{e^\tau_{j,k} : j,k = 0,1,2,\cdots\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$.

The Heat Kernel of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$

Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for all $f \in L^2(\mathbb{R}^2)$,

$$e^{-\rho L_\tau} f = \sum_{k=0}^\infty \sum_{j=0}^\infty e^{-(2k+1)|\tau|\rho} (f, e^\tau_{j,k}) e^\tau_{j,k}, \quad \rho > 0.$$

So, for $\rho > 0$,

$$e^{-\rho L_\tau} f = \sum_{k=0}^\infty e^{-(2k+1)|\tau|\rho} \sum_{j=0}^\infty (f, e^\tau_{j,k}) e^\tau_{j,k}.$$

To compute $\sum_{j=0}^\infty (f, e^\tau_{j,k}) e^\tau_{j,k}$, note that for $k = 0,1,2,\cdots$,

$$\begin{aligned}
f *_{\tau/4} e^\tau_{k,k}
&= \sum_{j=0}^\infty \sum_{l=0}^\infty (f, e^\tau_{j,l})(e^\tau_{j,l} *_{\tau/4} e^\tau_{k,k}) \\
&= \sum_{j=0}^\infty \sum_{l=0}^\infty (f, e^\tau_{j,l})(2\pi)^{1/2}|\tau|^{-1/2}\delta_{l,k} e^\tau_{j,k} \\
&= (2\pi)^{1/2}|\tau|^{-1/2} \sum_{j=0}^\infty (f, e^\tau_{j,k}) e^\tau_{j,k}.
\end{aligned}$$

So, for $k=0,1,2,\ldots$,

$$\sum_{j=0}^{\infty}(f,e^{\tau}_{j,k})e^{\tau}_{j,k}=(2\pi)^{-1/2}|\tau|^{1/2}\left(f*_{\tau/4}e^{\tau}_{k,k}\right).$$

$\therefore$ for $\rho>0$,

$$e^{-\rho L_\tau}f=(2\pi)^{-1/2}|\tau|^{1/2}\sum_{k=0}^{\infty}e^{-(2k+1)|\tau|\rho}\left(e^{\tau}_{k,k}*_{-\frac{\tau}{4}}f\right).$$

But

$$(2\pi)^{-1/2}|\tau|^{1/2}\sum_{k=0}^{\infty}e^{-(2k+1)|\tau|\rho}\,e^{\tau}_{k,k}(q,p)$$

$$=(2\pi)^{-1/2}|\tau|^{\cancel{1/2}\,1}e^{-|\tau|\rho}\sum_{k=0}^{\infty}e^{-2k|\tau|\rho}\,e_{k,k}\left(\frac{\tau}{\sqrt{|\tau|}}q,\sqrt{|\tau|}\,p\right)$$

$$\underset{\text{Mehler}}{=}(2\pi)^{-1}|\tau|e^{-|\tau|\rho}\frac{1}{1-e^{-2|\tau|\rho}}e^{-|\tau||z|^2\frac{1}{4}\frac{1+e^{-2|\tau|\rho}}{1-e^{-2|\tau|\rho}}}$$

$$=\frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)}e^{-\frac{1}{4}\tau|z|^2\coth(\tau\rho)}.$$

$\therefore$

$$\left(e^{-\rho L_\tau}f\right)(z)=\int_{\mathbb{C}}\frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)}e^{-\frac{1}{4}\tau|z-w|^2\coth(\tau\rho)}e^{-i\frac{\tau}{4}[z,w]}f(w)\,dw.$$

$\therefore$ the kernel K^{τ}_{ρ} of $e^{-\rho L_\tau}$ is given by

$$K^{\tau}_{\rho}(z,w)=\frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)}e^{-\frac{\tau}{4}|z-w|^2\coth(\tau\rho)}e^{-i\frac{\tau}{4}[z,w]}$$

for all $z,w\in\mathbb{C}$.