Lecture 20

Mehler's Formula for Hermite Functions on $\mathbb{R}^2$

Theorem: For all $z = q + ip$ in $\mathbb{C}$,

$$\sum_{k=0}^{\infty} e_{k,k}(z) r^k = (2\pi)^{-1/2} \frac{1}{1-r} e^{-\frac{1}{4}|z|^2 \frac{1+r}{1-r}}, \quad |r| < 1,$$

and the convergence of the series is absolute and uniform on compact subsets of $(-1,1)$.

Proof: For all y and p in $\mathbb{R}$, and all $r \in (-1,1)$, by Mehler's formula studied yesterday,

$$\sum_{k=0}^{\infty} e_k\left(y+\frac{p}{2}\right) e_k\left(y-\frac{p}{2}\right) r^k$$

$$= \frac{1}{\sqrt{\pi}}(1-r^2)^{-1/2} e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}\left(2y^2+\frac{p^2}{2}\right)+\frac{2r}{1-r^2}\left(y^2-\frac{p^2}{4}\right)}$$

$$= \frac{1}{\sqrt{\pi}}(1-r^2)^{-1/2} e^{-\frac{1+r^2}{1-r^2}y^2 - \frac{1+r^2}{1-r^2}\frac{p^2}{4} + \frac{2r}{1-r^2}y^2 - \frac{2r}{1-r^2}\frac{p^2}{4}}$$

$$= \frac{1}{\sqrt{\pi}}(1-r^2)^{-1/2} e^{-\frac{1-2r+r^2}{1-r^2}y^2 - \frac{1+2r+r^2}{1-r^2}\frac{p^2}{4}}$$

$$= \frac{1}{\sqrt{\pi}}(1-r^2)^{-1/2} e^{-\frac{1-r}{1+r}y^2 - \frac{1+r}{1-r}\frac{p^2}{4}}.$$

Taking the inverse Fourier transform of the preceding formula with respect to y,

we get

$$\sum_{k=0}^{\infty}\left[(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iyq}\, e_k\left(y+\frac{p}{2}\right) e_k\left(y-\frac{p}{2}\right) dy\right] r^k$$

$$= \frac{1}{\sqrt{\pi}}\left(1-r^2\right)^{-1/2} e^{-\frac{1+r}{1-r}\frac{p^2}{4}} (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iyq}\, e^{-\frac{1-r}{1+r}y^2}\, dy.$$

Let $y=-x$. Then

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iyq}\, e^{-\frac{1-r}{1+r}y^2}\, dy$$

$$=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ixq}\, e^{-\frac{2(1-r)}{1+r}\frac{x^2}{2}}\, dx \quad \left(a=\sqrt{\frac{2(1-r)}{1+r}}\right)$$

$$=\left(\mathcal{F}\left(D_a\varphi\right)\right)(q), \text{ where } \varphi(x)=e^{-\frac{x^2}{2}}, x\in\mathbb{R}.$$

$\therefore$ $$(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iyq}\, e^{-\frac{1-r}{1+r}y^2}\, dy$$

$$=a^{-1}\hat{\varphi}\left(\frac{q}{a}\right)$$

$$=\sqrt{\frac{1+r}{2(1-r)}}\, e^{-\frac{1}{2}q^2\frac{1+r}{2(1-r)}}.$$

$\therefore$ $$\sum_{k=0}^{\infty} e_{k,k}(z)\, r^k = \frac{1}{\sqrt{\pi}}\left(1-r^2\right)^{-1/2} e^{-\frac{1+r}{1-r}\frac{p^2}{4}} \sqrt{\frac{1+r}{2(1-r)}}\, e^{-\frac{q^2}{4}\frac{1+r}{1-r}}.$$

or

$$\sum_{k=0}^{\infty} e_{k,k}(z)\, r^k$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{(1+r)^{1/2}(1-r)^{1/2}} \frac{1}{\sqrt{2}} \frac{(1+r)^{1/2}}{(1-r)^{1/2}} e^{-(q^2+p^2)\frac{1+r}{1-r}\frac{1}{4}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1-r} e^{-\frac{1}{4}|z|^2}.$$