Lecture 19

$\tau/4$ - Twisted Convolutions of Hermite Functions on $\mathbb{R}^2$

Theorem: Let $\tau \in \mathbb{R} \setminus \{0\}$. Then for all nonnegative integers α, β, μ and ν,

$$e^{\tau}_{\alpha,\beta} *_{\tau/4} e^{\tau}_{\mu,\nu} = (2\pi)^{1/2} |\tau|^{-1/2} \delta_{\beta,\mu} e^{\tau}_{\alpha,\nu}.$$

Proof: Let $z = (q, p)$ and $w = (x, \xi)$ be points in $\mathbb{C}$. Then

$$\left(e^{\tau}_{\alpha,\beta} *_{\tau/4} e^{\tau}_{\mu,\nu}\right)(z)$$

$$= \int_{\mathbb{C}} e^{\tau}_{\alpha,\beta}(z-w)\, e^{\tau}_{\mu,\nu}(w)\, e^{i\frac{\tau}{2}[z,w]}\, dw$$

$$= |\tau| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e_{\alpha,\beta}\left(\frac{\tau}{\sqrt{|\tau|}}(q-x), \sqrt{|\tau|}(p-\xi)\right) e_{\mu,\nu}\left(\frac{\tau}{\sqrt{|\tau|}}x, \sqrt{|\tau|}\xi\right) e^{i\frac{\tau}{2}(xp-\xi q)}\, dx\, d\xi.$$

Let $q' = \frac{\tau}{\sqrt{|\tau|}} x$ and $p' = \sqrt{|\tau|}\,\xi$. Then

$$\begin{cases} dx = \frac{1}{\sqrt{|\tau|}} dq', \\ d\xi = \frac{1}{\sqrt{|\tau|}} dp'. \end{cases}$$

$$\therefore \left(e^{\tau}_{\alpha,\beta} *_{\tau/4} e^{\tau}_{\mu,\nu}\right)(z)$$

$$= \cancel{|\tau|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e_{\alpha,\beta}\left(\frac{\tau}{\sqrt{|\tau|}} q - q', \sqrt{|\tau|}\, p - p'\right) e_{\mu,\nu}(q', p')\, e^{i\frac{\tau}{2}\left(\frac{\sqrt{|\tau|}}{\tau} q' p - \frac{1}{\sqrt{|\tau|}} p' q\right)}\, dq'\, dp'$$

$$\therefore \left(e^{\tau}_{\alpha,\beta} *_{\tau/4} e^{\tau}_{\mu,\nu}\right)(z)$$

$$= |\tau|^{-1/2}|\tau|^{1/2}\left(e_{\alpha,\beta} *_{1/4} e_{\mu,\nu}\right)\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}p\right)$$

$$= (2\pi)^{1/2}|\tau|^{-1/2}\delta_{\beta,\mu}|\tau|^{1/2} e_{\alpha,\nu}\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}p\right)$$

$$= (2\pi)^{1/2}|\tau|^{-1/2}\delta_{\beta,\mu} e^{\tau}_{\alpha,\nu}(q,p).$$

Mehler's Formulas for Hermite Functions on $\mathbb{R}$

Theorem: For all x and y in $\mathbb{R}$, and all $w \in \mathbb{C}$ with $|w|<1$,

$$\sum_{k=0}^{\infty} e_k(x)\, e_k(y)\, w^k = \frac{1}{\sqrt{\pi}}(1-w^2)^{-1/2} e^{-\frac{1}{2}\frac{1+w^2}{1-w^2}(x^2+y^2)+\frac{2w}{1-w^2}xy},$$

where the series is uniformly and absolutely convergent on every compact subset of $\{w \in \mathbb{C} : |w|<1\}$.

Remarks: To be specific, we use the principal branch of $(1-w^2)^{-1/2}$, i.e.,

$$(1-w^2)^{-1/2} = e^{-\frac{1}{2}\mathrm{Log}(1-w^2)},$$

where

$$\mathrm{Log}\,\zeta = \ln|\zeta| + i\,\mathrm{Arg}\,\zeta, \quad -\pi < \mathrm{Arg}\,\zeta < \pi$$

$\therefore (1-w^2)^{-1/2}$ is holomorphic on $\{w \in \mathbb{C} : |w| < 1\}$ and for any $w \in \mathbb{C}$ with $|w| < 1$, we get

$$(1-w^2)^{-1/2} = e^{-\frac{1}{2}\ln|1-w^2|} > 0.$$