## Lecture 9

### The Initial Value Problem for the Heat Equation for $\mathcal{L}$:

Let $f$ be a given function on $\mathbb{H}^1$. Consider

$$\begin{cases} \frac{\partial u}{\partial \rho}(z,t,\rho) = -(\mathcal{L}u)(z,t,\rho), & (z,t)\in\mathbb{H}^1,\ \rho>0, \\ u(z,t,0) = f(z,t), & (z,t)\in\mathbb{H}^1. \end{cases}$$

Then formally,

$$u(z,t,\rho) = (e^{-\rho\mathcal{L}}f)(z,t),\ (z,t)\in\mathbb{H}^1,\ \rho>0,$$

and

$$e^{-\rho\mathcal{L}}f = f *_{\mathbb{H}^1} K_\rho,$$

where $K_\rho$ is the heat kernel of $\mathcal{L}$ on $\mathbb{H}^1$. Then for all $\tau\in\mathbb{R}$,

$$(e^{-\rho\mathcal{L}}f)^\tau = e^{-\rho L_\tau} f^\tau$$

and

$$(e^{-\rho\mathcal{L}}f)^\tau = (f * K_\rho)^\tau = (2\pi)^{1/2}(f^\tau *_{\tau/4} K_\rho^\tau)$$
$$= (2\pi)^{1/2}(K_\rho^\tau *_{-\frac{\tau}{4}} f^\tau).$$

So, if we can find the heat kernel $K_\rho^\tau$ of $e^{-\rho L_\tau}$ given by

$$k_p^\tau(z,w) = (2\pi)^{1/2} K_p^\tau(z) e^{-i\frac{\tau}{4}[z,w]}, \quad z, w \in \mathbb{C},$$

of $e^{-pL_\tau}$, then for all suitable $f$ on $\mathbb{H}'$,

$$(e^{-p\mathcal{L}} f)^\tau(z) = (2\pi)^{1/2} \int_{\mathbb{C}} K_p^\tau(z-w) f^\tau(w) e^{-i\frac{\tau}{4}[z,w]} dw$$

$$= (2\pi)^{1/2} (K_p *_{-\tau/4} f^\tau)(z) = (f *_{\mathbb{H}'} K_p)^\tau(z), \quad z \in \mathbb{C}.$$

$\therefore$ $e^{-p\mathcal{L}} f = f *_{\mathbb{H}'} K_p$ for all suitable $f$ on $\mathbb{H}'$.

The Poisson Equation : Let $f$ be a given function on $\mathbb{H}'$. Consider the equation $\mathcal{L}u = f$ on $\mathbb{H}'$. Then formally $u = \mathcal{L}^{-1} f$. We want to find a function $g$ on $\mathbb{H}'$ such that

$$\mathcal{L}^{-1} f = f *_{\mathbb{H}'} g.$$

Then for all $\tau \in \mathbb{R}$,

$$(\mathcal{L}^{-1} f)^\tau = (f *_{\mathbb{H}'} g)^\tau = (2\pi)^{1/2} (f^\tau *_{\tau/4} g^\tau) = (2\pi)^{1/2} (g^\tau *_{-\tau/4} f^\tau).$$

So, if we can find the Green function $G_\tau$ of $L_\tau$ given by

$$G_\tau(z,w) = (2\pi)^{1/2} g^\tau(z) e^{-i\frac{\tau}{4}[z,w]}, \quad z, w \in \mathbb{C},$$

then for all suitable $f$ on $\mathbb{H}'$,

$$(\mathcal{L}^{-1} f)^\tau(z) = (2\pi)^{1/2} \int_{\mathbb{C}} g^\tau(z-w) f^\tau(w) e^{-i\frac{\tau}{4}[z,w]} dw.$$

$\therefore$ for all suitable $f$ on $\mathbb{H}^1$,

$$(\mathcal{L}^{-1} f)^{\tau}(z) = (f *_{\mathbb{H}^1} g)^{\tau}(z), \quad z \in \mathbb{C}$$

$$\therefore \mathcal{L}^{-1} f = f *_{\mathbb{H}^1} g.$$

So, our tasks are to find the heat kernels and Green functions of $L_\tau$, $\tau \in \mathbb{R} \setminus \{0\}$. To do these, we need pseudo-differential operators via Wigner transforms and Weyl transforms.