Lecture 8

Consider the initial value problem for the heat equation driven by $\mathcal{L}$:

$$\begin{cases} \frac{\partial u}{\partial \rho}(z,t,\rho) = -(\mathcal{L}u)(z,t,\rho), & (z,t)\in \mathbb{H}^1,\ \rho>0, \\ u(z,t,0) = f(z,t), & (z,t)\in \mathbb{H}^1, \end{cases}$$

where f is a given function on $\mathbb{H}^1$. Formally,

$$u(z,t,\rho) = (e^{-\rho\mathcal{L}}f)(z,t),\ (z,t)\in\mathbb{H}^1,\ \rho>0.$$

A related equation is the Poisson equation given by

$$\mathcal{L}u = f,$$

where f is a given function on $\mathbb{H}^1$. A formal solution u is given by

$$u = \mathcal{L}^{-1}f.$$

Convolutions on $\mathbb{H}^1$: Let f and g be measurable functions on $\mathbb{H}^1$. Then we define the convolution $f *_{\mathbb{H}^1} g$ of f and g on $\mathbb{H}^1$ by

$$(f *_{\mathbb{H}^1} g)(z,t) = \int_{-\infty}^{\infty}\int_{\mathbb{C}} f((z,t)\cdot(w,s)^{-1})\,g(w,s)\,dw\,ds$$

for all $(z,t)\in\mathbb{H}^1$.

Aims: Find a function K_ρ, $\rho>0$, on $\mathbb{H}^1$ such that
$$e^{-\rho\mathcal{L}}f = f *_{\mathbb{H}^1} K_\rho, \ \rho>0.$$
We call K_ρ, $\rho>0$, the heat kernel of $\mathcal{L}$.

Find a function $\mathcal{G}$ on $\mathbb{H}^1$ such that
$$\mathcal{L}^{-1}f = f *_{\mathbb{H}^1} \mathcal{G}.$$
We call $\mathcal{G}$ the Green function of $\mathcal{L}$.

Convolutions on $\mathbb{H}^1$ and Twisted Convolutions:

Let $\lambda\in\mathbb{R}$. Then for all measurable functions f and g on $\mathbb{C}$, we define the twisted convolution $f *_\lambda g$ of f and g on $\mathbb{C}$ by
$$(f *_\lambda g)(z) = \int_{\mathbb{C}} f(z-w)g(w)e^{i\lambda[z,w]}\,dw, \ z\in\mathbb{C}.$$

Theorem: Let $f, g\in L^1(\mathbb{H}^1)$. Then
$$(f *_{\mathbb{H}^1} g)^\tau = (2\pi)^{1/2}(f^\tau *_{\tau/4} g^\tau).$$

Proof: For all $z\in\mathbb{C}$,
$$(f *_{\mathbb{H}^1} g)^\tau(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\tau}(f *_{\mathbb{H}^1} g)(z,t)\,dt$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\tau}\left(\int_{-\infty}^{\infty}\int_{\mathbb{C}} f((z,t)\cdot(w,s)^{-1})g(w,s)\,dw\,ds\right)dt.$$

$\therefore$ for all $z \in \mathbb{C}$,

$$(f *_{\mathbb{H}^1} g)^\tau(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\tau}\left(\int_{-\infty}^{\infty}\int_{\mathbb{C}} f(z-w, t-s-\tfrac{1}{4}[z,w])\, g(w,s)\,dw\,ds\right)dt$$

Let $t' = t - \frac{1}{4}[z,w]$. Then

$$(f *_{\mathbb{H}^1} g)^\tau(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{C}} e^{it'\tau} f(z-w, t'-s)\, g(w,s)\, e^{i\frac{\tau}{4}[z,w]}\,dw\,ds\,dt'.$$

On the other hand, we get

$$(f^\tau *_{\tau/4} g^\tau)(z)$$
$$= \int_{\mathbb{C}} f^\tau(z-w)\, g^\tau(w)\, e^{i\frac{\tau}{4}[z,w]}\,dw$$
$$= (2\pi)^{-1/2}\int_{\mathbb{C}} \left(\int_{-\infty}^{\infty} f(z-w, \cdot - s)\, g(w,s)\,ds\right)^{\vee}(\tau)\, e^{i\frac{\tau}{4}[z,w]}\,dw$$
$$= (2\pi)^{-1}\int_{\mathbb{C}}\int_{-\infty}^{\infty} e^{it\tau}\left(\int_{-\infty}^{\infty} f(z-w, t-s)\, g(w,s)\,ds\right) e^{i\frac{\tau}{4}[z,w]}\,dw\,dt.$$

So, by Fubini's theorem, the proof is complete.

<u>Remarks</u>: Let f be a suitable function on $\mathbb{H}^1$ and let $\rho > 0$.

Then

$$(e^{-\rho\mathcal{L}} f)^\tau = (f *_{\mathbb{H}^1} K_\rho)^\tau = (2\pi)^{1/2}(f^\tau *_{\tau/4} K_\rho^\tau) = (2\pi)^{1/2}(K_\rho^\tau *_{-\tau/4} f^\tau).$$

So, if we can compute the heat kernel $e^{-\frac{\tau}{4}[z,w]} K_\rho^\tau(z,w)$, $z, w \in \mathbb{C}$, $\rho > 0$, of L_τ, then the heat kernel K_ρ of $\mathcal{L}$ can be obtained by the Fourier transform of K_ρ^τ with respect to τ. Similar arguments hold for computing the Green function of $\mathcal{L}$ on $\mathbb{H}^1$.