Lecture 6

The Sub-Laplacian: $\mathcal{L} = -(X^2+Y^2)$.

$$\mathcal{L} = -\Delta - \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}.$$

Twisted Laplacians: Let $\tau \in \mathbb{R}\setminus\{0\}$. Then

$$L_\tau = -\Delta - \frac{1}{4}(x^2+y^2)\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau.$$

How are they connected?

Definition: Let f be a Schwartz function on $\mathbb{H}^1$. Then for all $\tau \in \mathbb{R}\setminus\{0\}$, we define the $\overbrace{\text{function}}^{f^\tau}$ on $\mathbb{C}$ by

$$f^\tau(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\tau} f(t)\,dt, \quad z \in \mathbb{C}.$$

Theorem: Let φ be a Schwartz function on $\mathbb{H}^1$. Then for all $\tau \in \mathbb{R}\setminus\{0\}$,

$$(\mathcal{L}\varphi)^\tau(z) = (L_\tau \varphi)(z), \quad z \in \mathbb{C}.$$

Proof: To begin with, let f be a Schwartz function on $\mathbb{R}$. Then

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{it\tau} f'(t)\,dt = (2\pi)^{-1/2}\left\{ e^{it\tau} f(t)\Big|_{-\infty}^{\infty} - i\tau\int_{-\infty}^{\infty} e^{it\tau} f(t)\,dt \right\}$$
$$= -i\tau\,\check{f}(\tau).$$

So, for all $(z,t) \in \mathbb{H}^1$,

$$(\mathcal{L}\varphi)(z,t) = (-\Delta\varphi)(z,t) - \frac{1}{4}(x^2+y^2)\frac{\partial^2\varphi}{\partial t^2}(z,t) + \left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right)(z,t).$$

$\therefore$ for all $(z,\tau) \in \mathbb{C} \times (\mathbb{R}\setminus\{0\})$,

$$(\mathcal{L}\varphi)^{\vee}(z,\tau) = (-\Delta\varphi)^{\vee}(z,\tau) + \frac{1}{4}(x^2+y^2)\tau^2\check{\varphi}(z,\tau) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau\check{\varphi}(z,\tau).$$

$\therefore$

$$(\mathcal{L}\varphi)^{\tau}(z) = (-\Delta\varphi)^{\tau}(z) + \frac{1}{4}(x^2+y^2)\tau^2\varphi^{\tau}(z) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau\varphi^{\tau}(z)$$

$$= (-\Delta\varphi^{\tau})(z) + \frac{1}{4}(x^2+y^2)\tau^2\varphi^{\tau}(z) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau\varphi^{\tau}(z).$$

$\therefore (\mathcal{L}\varphi)^{\tau} = L_{\tau}\varphi.$

Why do we reduced to L_{τ}, $\tau \in \mathbb{R}\setminus\{0\}$?

Need a general discussion: We need some notions from Linear Partial Differential Operators.

$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_j \in \mathbb{R}, j = 1, 2, \cdots, n\}$.

Points in $\mathbb{R}^n$ are denoted by $x, y, \cdots; \xi, \eta, \cdots$, etc.

Let $x, y \in \mathbb{R}^n$. Then we define $x \cdot y$ by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j \text{ (inner product of } x, y).$$

$$|x| = \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}} \text{ (norm of } x)$$

The simplest partial differential operators on $\mathbb{R}^n$ are

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$$

often denoted by

$$\partial_1, \partial_2, \dots, \partial_n.$$

Define D_j for $j = 1, 2, \cdots, n$ by

$$D_j = -i\, \partial_j.$$

The most general linear partial differential operator of order m on $\mathbb{R}^n$ is

$$\sum_{\alpha_1+\alpha_2+\cdots+\alpha_n \leq m} a_{\alpha_1, \alpha_2, \cdots, \alpha_n}(x) D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}. \quad (*)$$

To simplify the notation in $(*)$, let

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \text{ a multi-index.}$$

Let

$$|\alpha| = \sum_{j=1}^{n} \alpha_j, \text{ length of } \alpha.$$

Let α be a multi-index. Then we define D^α by

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}.$$

Then (*) becomes

$$\sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

often denoted by $P(x, D)$.

For all $\xi \in \mathbb{R}^n$, we also define ξ^α by

$$\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}.$$

Replacing D by ξ, we get

$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$, which is called the symbol of $P(x, D)$.

<u>Definition</u>: $P_m(x, \xi)$, given by $\sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$, is called the <u>principal symbol</u> of $P(x, D)$.