Lecture 5

Recall: $[X, Y] = -T \Rightarrow X, Y$ and their first-order commutators span $\mathfrak{h}^1$. $\therefore X, Y$ alone determine $\mathfrak{h}^1$. They are known as the horizontal vector fields on $\mathbb{H}^1$ and T is known as the "missing" direction.

Definition: The sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^1$ is defined by
$$\mathcal{L} = -(X^2 + Y^2).$$

To see the sub-Laplacian more explicitly,
$$X^2 = \left(\frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t}\right)$$
$$= \frac{\partial^2}{\partial x^2} + \frac{1}{2} y \frac{\partial^2}{\partial x \partial t} + \frac{1}{2} y \frac{\partial^2}{\partial t \partial x} + \frac{1}{4} y^2 \frac{\partial^2}{\partial t^2}$$
$$= \frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial x \partial t} + \frac{1}{4} y^2 \frac{\partial^2}{\partial t^2}.$$
$$Y^2 = \left(\frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial t}\right)$$
$$= \frac{\partial^2}{\partial y^2} - \frac{1}{2} x \frac{\partial^2}{\partial y \partial t} - \frac{1}{2} x \frac{\partial^2}{\partial t \partial y} + \frac{1}{4} x^2 \frac{\partial^2}{\partial t^2}.$$
$\therefore X^2 + Y^2$
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} - \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}.$$
$$\therefore \mathcal{L} = -\Delta - \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} + \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}.$$

Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be linear partial differential operators on $\mathbb{R}^2$ given by

$$\begin{cases} \frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \end{cases}$$

Then we define for $\tau \in \mathbb{R}\setminus\{0\}$, the linear partial differential operators Z_τ and $\bar{Z}_\tau$ by

$$\begin{cases} Z_\tau = \frac{\partial}{\partial z} + \frac{1}{2}\tau\bar{z}, \quad \bar{z} = x - iy, \\ \bar{Z}_\tau = \frac{\partial}{\partial \bar{z}} - \frac{1}{2}\tau z, \quad z = x + iy. \end{cases}$$

In fact, $-\bar{Z}_\tau$ is the formal adjoint of Z_τ in the sense that

$$(Z_\tau \varphi, \psi) = -(\varphi, \bar{Z}_\tau \psi)$$

for all Schwartz functions on $\mathbb{H}^1$. Let L_τ be the linear partial differential operator on $\mathbb{R}^2$ defined by

$$L_\tau = -\frac{1}{2}(Z_\tau\bar{Z}_\tau + \bar{Z}_\tau Z_\tau).$$

Let us write down L_τ explicitly.

$$Z_\tau \bar{Z}_\tau = \left(\frac{\partial}{\partial z} + \frac{1}{2}\tau \bar{z} \right)\left(\frac{\partial}{\partial \bar{z}} - \frac{1}{2}\tau z \right)$$

$$= \Delta - \frac{1}{2}\tau z \frac{\partial}{\partial z} - \frac{1}{2}\tau + \frac{1}{2}\tau \bar{z} \frac{\partial}{\partial \bar{z}} - \frac{1}{4}\tau^2 |z|^2,$$

and

$$\bar{Z}_\tau Z_\tau = \left(\frac{\partial}{\partial \bar{z}} - \frac{1}{2}\tau z \right)\left(\frac{\partial}{\partial z} + \frac{1}{2}\tau \bar{z} \right)$$

$$= \Delta + \frac{1}{2}\tau \bar{z} \frac{\partial}{\partial \bar{z}} + \frac{1}{2}\tau - \frac{1}{2}\tau z \frac{\partial}{\partial z} - \frac{1}{4}\tau^2 |z|^2.$$

$$\therefore L_\tau = -\frac{1}{2}\left(Z_\tau \bar{Z}_\tau + \bar{Z}_\tau Z_\tau \right)$$

$$= -\frac{1}{2}\left(2\Delta + \tau \bar{z} \frac{\partial}{\partial \bar{z}} - \tau z \frac{\partial}{\partial z} - \frac{1}{2}\tau^2 |z|^2 \right)$$

Now,

$$\bar{z} \frac{\partial}{\partial \bar{z}} = (x - iy)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right)$$

$$= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right).$$

and

$$z \frac{\partial}{\partial z} = (x + iy)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right)$$

$$= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

$$\therefore L_\tau = \underbrace{-\Delta + \frac{1}{4}(x^2 + y^2)}_{\substack{\text{Hermite operator} \\ \text{or Simple harmonic} \\ \text{oscillator}}} - i\underbrace{\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)}_{\substack{\text{rolation or} \\ \text{angular momentum operator}}}$$