Lecture 4

Recall: Let $\varepsilon = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t} \in T_{(0,0,0)}\mathbb{H}'$, where

Then we get

$$V^{\varepsilon} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \left(c + \frac{1}{2}ay - \frac{1}{2}bx\right)\frac{\partial}{\partial t}.$$

Now, $V^{\varepsilon} \in h'$. Indeed, let $(w,s) = (u,v,s) \in \mathbb{H}'$. Then for all $(z,t) = (x,y,t) \in \mathbb{H}'$, $f \in C^{\infty}(\mathbb{H}')$,

$$\left(V^{\varepsilon}L_{(w,s)}f\right)(z,t) = V^{\varepsilon}\Big(f\big(\underbrace{u+x}_{\xi}, \underbrace{v+y}_{\eta}, \underbrace{s+t+\tfrac{1}{2}(vx-uy)}_{\tau}\big)\Big)$$

$$= \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \left(c + \frac{1}{2}ay - \frac{1}{2}bx\right)\frac{\partial}{\partial t}\right)\left(f(\cdots)\right)$$

$$= a\frac{\partial f}{\partial x}(\cdots) + \frac{1}{2}av\frac{\partial f}{\partial t}(\cdots) + b\frac{\partial f}{\partial y}(\cdots) - \frac{1}{2}bu\frac{\partial f}{\partial t}(\cdots)$$
$$+ \left(c + \frac{1}{2}ay - \frac{1}{2}bx\right)\frac{\partial f}{\partial t}(\cdots)$$

$$= a\frac{\partial f}{\partial x}(\cdots) + b\frac{\partial f}{\partial y}(\cdots) + \left(c + \frac{1}{2}a(v+y) - \frac{1}{2}b(u+x)\right)\frac{\partial f}{\partial t}(\cdots)$$

Next,

$$\left(L_{(w,s)}V^{\varepsilon}f\right)(z,t) = \left(V^{\varepsilon}f\right)(\cdots)$$

$$= \left[\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \left(c + \frac{1}{2}ay - \frac{1}{2}bx\right)\frac{\partial}{\partial t}\right)f\right](\cdots)$$

$$= a\frac{\partial f}{\partial x}(\cdots) + b\frac{\partial f}{\partial y}(\cdots) + \left(c + \frac{1}{2}a(v+y) - \frac{1}{2}b(u+x)\right)\frac{\partial f}{\partial t}(\cdots).$$

$\therefore V^{\varepsilon}L_{(w,s)} = L_{(w,s)}V^{\varepsilon}$. $\therefore V^{\varepsilon} \in h'$.

It is obvious that

$$V^{\varepsilon_1} + V^{\varepsilon_2} = V^{\varepsilon_1+\varepsilon_2}, \quad \varepsilon_1, \varepsilon_2 \in T_{(0,0,0)}\mathbb{H}',$$

and

$$\alpha V^{\varepsilon} = V^{\alpha\varepsilon}, \quad \alpha \in \mathbb{R}, \ \varepsilon \in T_{(0,0,0)}\mathbb{H}'.$$

$\therefore T_{(0,0,0)}\mathbb{H}'$ and $\mathfrak{h}'$ are isomorphic vector spaces over $\mathbb{R}$. $\therefore \mathfrak{h}'$ is a three-dimensional real vector space. So, we only need to prove that X, Y and T are linearly independent. But for all $a, b, c \in \mathbb{R}$ with $aX + bY + cT = 0$, let f be the function on $\mathbb{H}'$ given by

$$f(x,y,t) = x, \quad (x,y,t) \in \mathbb{H}'.$$

Then

$$((aX + bY + cT)f)(x,y,t) = 0$$

$$\Leftrightarrow \left(\left[a\left(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}\right) + b\left(\frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t}\right) + c\frac{\partial}{\partial t}\right]f\right)(x) = 0$$

$$\Leftrightarrow a = 0.$$

Similarly, $b=0$ and $c=0$. $\therefore \{X, Y, T\}$ is a basis for $\mathfrak{h}'$.

The following theorem, which is an exercise tells us that X, Y and their <u>first</u> commutators span $\mathfrak{h}'$. Hence we call

$\mathbb{H}^1$ a step 1-Heisenberg group.

Theorem: $[X, Y] = -T$ and all other commutators are zero.

Remarks: $\{X, Y, T\}$ is a basis for $\mathfrak{h}^1$. So are many others. Why is $\{X, Y, T\}$ the preferred one?

Theorem: Let c_1, c_2 and c_3 be the coordinate axes in $\mathbb{H}^1$ and write them as

$$c_1(s) = (s, 0, 0),$$
$$c_2(s) = (0, s, 0),$$

and

$$c_3(s) = (0, 0, s)$$

for all $s \in \mathbb{R}$. Then for all C^∞ function on $\mathbb{H}^1$, $(z, t) \in \mathbb{H}^1$,

$$(Xf)(z,t) = \frac{d}{ds}\Big|_{s=0} f((z,t)\cdot c_1(s)),$$

$$(Yf)(z,t) = \frac{d}{ds}\Big|_{s=0} f((z,t)\cdot c_2(s)),$$

and

$$(Tf)(z,t) = \frac{d}{ds}\Big|_{s=0} f((z,t)\cdot c_3(s)).$$

Proof: (X only) Let $f \in C^\infty(\mathbb{H}^1)$. Then for all $(z,t) \in \mathbb{H}^1$,

$(Xf)(z,t)$

$$= \frac{d}{ds}\Big|_{s=0} f((z,t)\cdot c_1(s)) = \frac{d}{ds}\Big|_{s=0} f((x,y,t)\cdot(s,0,0))$$

$$= \frac{d}{ds}\Big|_{s=0} f\left(x+s, y, t+\tfrac{1}{2}sy\right)$$

$$= \left(\frac{\partial f}{\partial x}\left(x+s, y, t+\tfrac{1}{2}sy\right) + \frac{\partial f}{\partial t}\left(x+s, y, t+\tfrac{1}{2}sy\right)\left(\tfrac{1}{2}y\right)\right)\Big|_{s=0}$$

$$= \frac{\partial f}{\partial x}(x,y,t) + \frac{1}{2}y\frac{\partial f}{\partial t}(x,y,t).$$

$$\therefore\ X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}.$$