Lecture 3

Theorem: Let X, Y and T be vector fields on $\mathbb{H}^1$ given by

$$X = \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial t}$$

and

$$T = \frac{\partial}{\partial t}.$$

Then $\{X, Y, T\}$ is a basis for h^1.

Proof: We first prove that $X \in h^1$, i.e.,

$$X L_{(\omega,s)} = L_{(\omega,s)} X, \quad (\omega, s) \in \mathbb{H}^1.$$

Let $w = (u,v)$, $z = (x,y)$. Then for all $f \in C^\infty(\mathbb{H}^1)$,

$$\begin{aligned}(L_{(\omega,s)} f)(z,t) &= f((\omega,s)\cdot(z,t)) \\ &= f\left(u+x, v+y, s+t+\tfrac{1}{2}(vx-uy)\right), \\ &\qquad (z,t) \in \mathbb{H}^1.\end{aligned}$$

Let $(\cdots) = (\overset{\xi}{u+x}, \overset{\eta}{v+y}, \overset{\tau}{s+t+\tfrac{1}{2}(vx-uy)})$. Then

$$\begin{aligned}&(X L_{(\omega,s)} f)(z,t) \\ &= \left(\frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t}\right)\left[(L_{(\omega,s)} f)(z,t)\right] \\ &= \frac{\partial f}{\partial x}(\cdots) + \left(\tfrac{1}{2} v\right) \frac{\partial f}{\partial t}(\cdots) + \frac{1}{2} y \frac{\partial f}{\partial t}(\cdots).\end{aligned}$$

$$= \frac{\partial f}{\partial x}(\cdots) + \frac{1}{2}(v+y)\frac{\partial f}{\partial t}(\cdots).$$

Now,

$$(L_{(w,s)} X f)(z,t) = (Xf)\left(u+x, v+y, s+t+\tfrac{1}{2}(vx-uy)\right)$$

$$= \left(\frac{\partial f}{\partial x} + \frac{1}{2} y \frac{\partial f}{\partial t}\right)(\cdots)$$

$$= \frac{\partial f}{\partial x}(\cdots) + \frac{1}{2}(v+y)\frac{\partial f}{\partial t}(\cdots).$$

$\therefore X \in h_1$. Similarly, $Y, T \in h'$. Now, a basis for the tangent space $T_{(0,0,0)}\mathbb{H}'$ is $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right\}$.

We can identify h' and $T_{(0,0,0)}\mathbb{H}'$ as follows:

To every V in h' given by

$$V(x,y,t) = a(x,y,t)\frac{\partial}{\partial x} + b(x,y,t)\frac{\partial}{\partial y} + c(x,y,t)\frac{\partial}{\partial t},$$

we associate the tangent vector

$$V(0,0,0) = a(0,0,0)\frac{\partial}{\partial x} + b(0,0,0)\frac{\partial}{\partial y} + c(0,0,0)\frac{\partial}{\partial t}.$$

Conversely, given any tangent vector

$$\varepsilon = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t} \in T_{(0,0,0)}\mathbb{H}',$$

we let $\quad \gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)),\ s \in \mathbb{R},$

be any curve in $\mathbb{H}^1$ such that

$$\gamma(0) = (0,0,0),$$

$$\gamma'(0) = \varepsilon.$$

Let V^{ε} be the vector field on $\mathbb{H}^1$ defined by

$$(V^{\varepsilon}f)(x,y,t) = \frac{d}{ds}\bigg|_{s=0} f((x,y,t)\cdot\gamma(s)),\ (x,y,t) \in \mathbb{H}^1.$$

Hence for all $(x,y,t) \in \mathbb{H}^1$,

$(V^{\varepsilon}f)(x,y,t)$

$$= \frac{d}{ds}\bigg|_{s=0} f\left(x+\gamma_1(s), y+\gamma_2(s), t+\gamma_3(s)+t+\gamma_3(s)+\tfrac{1}{2}(y\gamma_1(s)-x\gamma_2(s)\right)$$

$$= \gamma_1'(0)\frac{\partial f}{\partial x}(x,y,t) + \gamma_2'(0)\frac{\partial f}{\partial y}(x,y,t) + \left(\gamma_3'(0)+\tfrac{1}{2}y\gamma_1'(0)-\tfrac{1}{2}x\gamma_2'(0)\right)\frac{\partial f}{\partial t}(x,y,t).$$

So,

$$(V^{\varepsilon}f)(x,y,t) = a\frac{\partial f}{\partial x}(x,y,t) + b\frac{\partial f}{\partial y}(x,y,t) + \left(c+\tfrac{1}{2}ay-\tfrac{1}{2}bx\right)\frac{\partial f}{\partial t}(x,y,t).$$

So, V^{ε} depends only on

$$\varepsilon = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t}.$$