Lecture 2

A Lie algebra $\mathfrak{g}$ is a real vector space on which there is a binary operation $[\,,\,]$ such that $[\,,\,]$ is bilinear and the Jacobi identity

$$[g_1,[g_2,g_3]]+[g_2,[g_3,g_1]]+[g_3,[g_1,g_2]]=0$$

is valid for all g_1, g_2, g_3 in $\mathfrak{g}$. We call $[\,,\,]$ the bracket in $\mathfrak{g}$. To give an example of a Lie algebra, we look at vector fields on $\mathbb{H}^1$. A vector field V on $\mathbb{H}^1$ is of the form

$$V(x,y,t)=a(x,y,t)\frac{\partial}{\partial x}+b(x,y,t)\frac{\partial}{\partial y}+c(x,y,t)\frac{\partial}{\partial t}$$

for all $(x,y,t)\in\mathbb{H}^1$, where a, b, c are C^∞ real-valued functions on $\mathbb{H}^1$.

Definition: A vector field V on $\mathbb{H}^1$ is said to be left-invariant if

$$V L_{(\omega,s)} = L_{(\omega,s)} V, \qquad (\omega,s)\in\mathbb{H}^1,$$

where $L_{(\omega,s)}$ is the left translation by (ω,s) given by

$$(L_{(\omega,s)} f)(z,t) = f((\omega,s)\cdot(z,t)), \quad (z,t)\in\mathbb{H}.$$

<u>Theorem:</u> Let $\mathfrak{h}'$ be the set of all left-invariant vector fields on $\mathbb{H}'$. Then $\mathfrak{h}'$ is a Lie algebra in which the bracket $[\ ,\ ]$ is the commutator given by

$$[X,Y] = XY - YX, \quad X, Y \in \mathfrak{h}'.$$

<u>Proof:</u> That $\mathfrak{h}'$ is a real vector space is easy to check. We first want to show that

$$[X,Y] \in \mathfrak{h}', \quad X, Y \in \mathfrak{h}'.$$

To do this, let

$$X = a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1 \frac{\partial}{\partial t},$$

$$Y = a_2 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + c_2 \frac{\partial}{\partial t},$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are C^∞ and real-valued functions on $\mathbb{H}'$. Then for all $f \in C^\infty(\mathbb{H}')$,

$$(XY)(f) = \left(a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1 \frac{\partial}{\partial t}\right)\left(a_2 \frac{\partial f}{\partial x} + b_2 \frac{\partial f}{\partial y} + c_2 \frac{\partial f}{\partial t}\right)$$

$$= a_1 a_2 \frac{\partial^2 f}{\partial x^2} + a_1 b_2 \frac{\partial^2 f}{\partial x \partial y} + a_1 c_2 \frac{\partial^2 f}{\partial x \partial t} +$$
$$b_1 c_2 \frac{\partial^2 f}{\partial y \partial x} + b_1 b_2 \frac{\partial^2 f}{\partial y^2} + b_1 c_2 \frac{\partial^2 f}{\partial y \partial t} +$$
$$c_1 a_2 \frac{\partial^2 f}{\partial t \partial x} + c_1 b_2 \frac{\partial^2 f}{\partial t \partial y} + c_1 c_2 \frac{\partial^2 f}{\partial t^2} + V_1,$$

where V_1 is a vector field on $\mathbb{H}'$. By switching subscripts in the second-order term in XY, we get

$$[X, Y] = XY - YX = V_1 - V_2 .$$

To see that $[X, Y]$ is left-invariant, let $(\omega, s) \in \mathbb{H}'$. Then

$$L_{(\omega,s)} XY = X L_{(\omega,s)} Y = XY L_{(\omega,s)}$$

and

$$L_{(\omega,s)} YX = Y L_{(\omega,s)} X = YX L_{(\omega,s)} .$$

$$\therefore \quad L_{(\omega,s)} [X, Y] = [X, Y] L_{(\omega,s)} .$$

$\therefore [X, Y] \in h'$.

That $[X, Y]$ is bilinear is easy to check. Finally, for all vector fields X, Y, Z on $\mathbb{H}'$,

$$[X, [Y, Z]] = [X, YZ - ZY]$$
$$= XYZ - XZY - YZX + ZYX,$$
$$[Y, [Z, X]] = [Y, ZX - XZ]$$
$$= YZX - YXZ - ZXY + XZY,$$
$$[Z, [X, Y]] = [Z, XY - YX]$$
$$= ZXY - ZYX - XYZ - YXZ,$$

$\therefore [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \overset{2.4}{=} 0$

So $\mathfrak{h}$ is a Lie algebra.