Lecture 18

Theorem: For all nonnegative integers α, β, μ, ν,

$$e_{\alpha,\beta} *_{1/4} e_{\mu,\nu} = (2\pi)^{1/2} \delta_{\beta,\mu} e_{\alpha,\nu},$$

where

$$\delta_{\beta,\mu} = \begin{cases} 1, & \beta = \mu, \\ 0, & \beta \neq \mu. \end{cases}$$

Proof: Let $\varphi, \psi \in \mathcal{S}$. Then

$$(W_{\widehat{e_{\alpha,\beta}}} \varphi, \psi)$$

$$= (2\pi)^{-1/2} \int_{\mathbb{C}} \widehat{e_{\alpha,\beta}}(z) W(\varphi,\psi)(z) dz$$

$$= (2\pi)^{-1/2} \int_{\mathbb{C}} W(e_\alpha, e_\beta)(z) W(\varphi,\psi)(z) dz$$

$$= (2\pi)^{-1/2} \int_{\mathbb{C}} \overline{W(e_\beta, e_\alpha)(z)} W(\varphi,\psi)(z) dz$$

$$= (2\pi)^{-1/2} (W(\varphi,\psi), W(e_\beta, e_\alpha))$$

$$= (2\pi)^{-1/2} (\varphi, e_\beta) \overline{(\psi, e_\alpha)}$$

$$= (2\pi)^{-1/2} (\varphi, e_\beta)(e_\alpha, \psi)$$

$$= (2\pi)^{-1/2} ((\varphi, e_\beta) e_\alpha, \psi).$$

$$\therefore W_{\widehat{e_{\alpha,\beta}}} \varphi = (2\pi)^{-1/2} (\varphi, e_\beta) e_\alpha.$$

Therefore from the preceding analysis, for all $\varphi \in \mathcal{S}$,

$$W_{\widehat{e_{\alpha,\beta}}} W_{\widehat{e_{\mu,\nu}}} \varphi$$

$$= (2\pi)^{-1/2} \left(W_{\widehat{e_{\mu,\nu}}} \varphi, e_\beta \right) e_\alpha$$

$$= (2\pi)^{-1} (\varphi, e_\nu)(e_\mu, e_\beta) e_\alpha$$

$$= (2\pi)^{-1} (\varphi, e_\nu) \delta_{\beta,\mu} e_\alpha$$

$$= (2\pi)^{-1/2} W_{\delta_{\beta,\mu} \widehat{e_{\alpha,\nu}}} \varphi.$$

But

$$W_{\widehat{e_{\alpha,\beta}}} W_{\widehat{e_{\mu,\nu}}} = W_\omega ,$$

where $\hat{\omega} = \left(\widehat{\widehat{e_{\alpha,\beta}}} *_{1/4} \widehat{\widehat{e_{\mu,\nu}}} \right) (2\pi)^{-1}$

$$= (2\pi)^{-1} \left(\widetilde{e_{\alpha,\beta}} *_{1/4} \widetilde{e_{\mu,\nu}} \right).$$

$$\therefore \quad (2\pi)^{-1/2} \delta_{\beta,\mu} \widetilde{e_{\alpha,\nu}} = (2\pi)^{-1} \widetilde{e_{\alpha,\beta} *_{\frac{1}{4}} e_{\mu,\nu}}$$

By Exercise 5 in Chapter 20,

$$\delta_{\beta,\mu} e_{\alpha,\nu} = (2\pi)^{-1/2} \left(e_{\alpha,\beta} *_{1/4} e_{\mu,\nu} \right).$$

The τ-Version of Twisted Convolutions of Hermite Functions on $\mathbb{R}^2$

Theorem: Let $\tau \in \mathbb{R} \setminus \{0\}$. Then for all nonnegative integers α, β, μ and ν,

$$e^{\tau}_{\alpha,\beta} *_{\tau/4} e^{\tau}_{\mu,\nu} = (2\pi)^{1/2} |\tau|^{-1/2} \delta_{\beta,\mu} e^{\tau}_{\alpha,\nu}.$$

Proof: Let $z = (q,p)$ and $w = (x,\xi)$ be points in $\mathbb{C}$. Then

$$\left(e^{\tau}_{\alpha,\beta} *_{\tau/4} e^{\tau}_{\mu,\nu}\right)(z)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\tau}_{\alpha,\beta}(z-w)\, e^{\tau}_{\mu,\nu}(w)\, e^{i\frac{\tau}{4}[z,w]}\, dw$$

$$= |\tau| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_{\alpha,\beta}\left(\frac{\tau}{\sqrt{|\tau|}}(q-x), \sqrt{|\tau|}(p-\xi)\right) e_{\mu,\nu}\left(\frac{\tau}{\sqrt{|\tau|}}x, \sqrt{|\tau|}\,\xi\right) e^{i\frac{\tau}{4}(xp-\xi q)}\, dx\, d\xi$$

Then we continue next time.