Lecture 17

Spectral Analysis of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$

Theorem: Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for $j,k = 0,1,2,\ldots$,

$$L_\tau e^\tau_{j,k} = (2k+1)|\tau| e^\tau_{j,k}.$$

Lemma: Let $x = \frac{\tau}{\sqrt{|\tau|}} q$, $y = \sqrt{|\tau|} p$. Then

$$L_\tau^{(q,p)} = |\tau| L^{(x,y)},$$

where

$$L_\tau^{(q,p)} = -\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right) + \frac{1}{4}(q^2+p^2)\tau^2 - i\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right)\tau$$

and

$$L^{(x,y)} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{4}(x^2+y^2) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Remark: In a suitable coordinate system, the twisted Laplacian L_τ, $\tau \in \mathbb{R}\setminus\{0\}$, is just $|\tau| L$.

Proof of Lemma:

$$\frac{\partial}{\partial q} = \frac{\partial}{\partial x}\frac{\tau}{\sqrt{|\tau|}}, \quad \frac{\partial}{\partial p} = \frac{\partial}{\partial y}\sqrt{|\tau|},$$

so,

$$\frac{\partial^2}{\partial q^2} = \frac{\partial^2}{\partial x^2}|\tau|, \quad \frac{\partial^2}{\partial p^2} = \frac{\partial^2}{\partial y^2}|\tau|.$$

$\therefore$

$$L_\tau^{(q,p)} = -|\tau|\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+|\tau|\frac{1}{4}(x^2+y^2)-|\tau| i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$$
$$= |\tau|\, L^{(x,y)}.$$

<u>Proof of Theorem on Spectral Analysis of L_τ, $\tau\in\mathbb{R}\setminus\{0\}$</u>

$$(L_\tau e^\tau_{j,k})(q,p)$$
$$= |\tau|^{1/2}(L_\tau^{(q,p)} e^\tau_{j,k})(q,p)$$
$$= |\tau|^{1/2}|\tau|(L^{(x,y)} e_{j,k})(x,y)$$
$$= |\tau|^{1/2}|\tau|(2k+1)\, e_{j,k}(x,y)$$
$$= |\tau|^{1/2}|\tau|(2k+1)\, e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}\,p\right)$$
$$= (2k+1)|\tau|\, e^\tau(q,p)$$

$$\therefore\ L_\tau e^\tau_{j,k} = (2k+1)|\tau|\, e^\tau_{j,k}.$$

<u>Remark</u>: $e^\tau_{j,k}$ is an eigenfunction of L_τ with corresponding eigenvalue $(2k+1)|\tau|$. Therefore $(2k+1)|\tau|$ is an eigenvalue of $e^\tau_{j,k}$ with infinite multiplicity.

Heat Kernels Related to the Heisenberg Group

Heat Kernels of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$

We first need some preparations:

Theorem: For all nonnegative integers α, β, μ and ν,

$$e_{\alpha,\beta} *_{\frac{1}{4}} e_{\mu,\nu} = (2\pi)^{1/2}\delta_{\beta,\nu}\, e_{\alpha,\nu}.$$

Remarks: ① The message here is that the twisted convolution of order $\frac{1}{4}$ of two Hermite functions on $\mathbb{R}^2$ is also a Hermite function on $\mathbb{R}^2$ up to a constant. Note that $e_{\alpha,\nu}$ is a Hermite function on $\mathbb{R}^2$.

②
$$\delta_{\beta,\nu} = \begin{cases} 1 & \text{if } \beta = \nu, \\ 0 & \text{if } \beta \neq \nu. \end{cases}$$