Lecture 16

Let $Z = \frac{\partial}{\partial z} + \frac{1}{2}\bar{z}$, $\bar{Z} = \frac{\partial}{\partial \bar{z}} - \frac{1}{2}z$, where

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Theorem: For all z in $\mathbb{C}$,

$(Z e_{j,k})(z) = i(2k)^{1/2} e_{j,k-1}(z)$, $j = 0,1,2,\ldots$, $k = 1,2,\ldots$,
and
$(\bar{Z} e_{j,k})(z) = i(2k+2)^{1/2} e_{j,k+1}(z)$, $j,k = 0,1,2,\ldots$.

Remarks: Z annihilates and $\bar{Z}$ creates states in quantum mechanics.

Recall: We call L the twisted Laplacian with $\tau = 1$.

Theorem: For all $j,k = 0,1,2,\ldots$,

$$L e_{j,k} = (2k+1) e_{j,k}$$

Theorem: Let $\tau \in \mathbb{R}\setminus\{0\}$. Then $\{e^{\tau}_{j,k} : j,k = 0,1,2,\ldots\}$ is an orthonormal basis for $L^2(\mathbb{C})$.

Proof: For nonnegative integers α, β, μ and ν

$$(e^{\tau}_{\alpha,\beta}, e^{\tau}_{\mu,\nu}) = |\tau| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_{\alpha,\beta}\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}p\right) \overline{e_{\mu,\nu}\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}p\right)}\, dq\,dp$$

for $j,k = 0,1,2,\ldots$.

Let $q' = \frac{\tau}{\sqrt{|\tau|}} q$ and $p' = \sqrt{|\tau|}\, p$. Then

$$(e^{\tau}_{\alpha,\beta}, e^{\tau}_{\mu,\nu})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e_{\alpha,\beta}(q', p')\overline{e_{\mu,\nu}(q', p')}\, dq'\, dp'.$$

Therefore $\{ e^{\tau}_{j,k} : j, k = 0, 1, 2, \dots \}$ is an orthonormal set in $L^2(\mathbb{C})$. Now, let $f \in L^2(\mathbb{C})$ be such that

$$(f, e^{\tau}_{j,k}) = 0, \quad j, k = 0, 1, 2, \dots.$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(q,p)\overline{e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}} q, \sqrt{|\tau|}\, p\right)}\, dq\, dp = 0$$

Let $q' = \frac{\tau}{\sqrt{|\tau|}} q$ and $p' = \sqrt{|\tau|}\, p$ be as above.

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\frac{\sqrt{|\tau|}}{\tau} q', \frac{1}{\sqrt{|\tau|}} p'\right)\overline{e_{j,k}(q', p')}\, dq'\, dp' = 0$$

But $\{ e_{j,k} : j, k = 0, 1, 2, \dots \}$ is an orthonormal basis for $L^2(\mathbb{C})$. Therefore

$$f\left(\frac{\sqrt{|\tau|}}{\tau} q', \frac{1}{\sqrt{|\tau|}} p'\right) = 0$$

for almost all $(q', p') \in \mathbb{R}^2$. Therefore $f(q', p') = 0$ for almost all (q', p') in $\mathbb{C}$. This completes the proof.

Complete Spectral Analysis of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$

Theorem: Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for all $j, k = 0, 1, 2, \cdots$,

$$L_\tau e^\tau_{j,k} = (2k+1)|\tau| e^\tau_{j,k}.$$

Remarks: For $j, k = 0, 1, 2, \cdots$,

$e^\tau_{j,k}$ is an eigenfunction of L_τ corresponding to the eigenvalue $(2k+1)|\tau|$. So, there are infinitely many eigenfunctions corresponding to the same eigenvalue $(2k+1)|\tau|$. We say that $(2k+1)|\tau|$ is an eigenvalue of L_τ with infinite multiplicity.

The proof of the theorem is by a change of coordinates. To be done next time (i.e., on Thursday, July 28, 2021).