Lecture 15

Recall: $e_{j,k} = V(e_j, e_k)$, $j,k = 0,1,2,\cdots$.

Theorem: $\{e_{j,k} : j,k = 0,1,2,\cdots\}$ is an orthonormal basis for $L^2(\mathbb{C})$.

Proof: For all nonnegative integers j_1, j_2, k_1, k_2,

$$(e_{j_1,k_1}, e_{j_2,k_2})$$
$$= (V(e_{j_1}, e_{k_1}), V(e_{j_2}, e_{k_2}))$$
$$= (e_{j_1}, e_{j_2})(e_{k_1}, e_{k_2}) = 0$$

unless $j_1 = j_2$ and $k_1 = k_2$. If $j_1 = j_2$ and $k_1 = k_2$, then $(e_{j_1,k_1}, e_{j_2,k_2}) = 1$. $\therefore$ $\{e_{j,k} : j,k = 0,1,2,\cdots\}$ is an orthonormal set in $L^2(\mathbb{R}^2)$. To complete the proof, we have to show that if $f \in L^2(\mathbb{R}^2)$ is such that

$$(f, e_{j,k}) = 0, \quad j,k = 0,1,2,\cdots,$$

then $f = 0$ almost everywhere on $\mathbb{R}^2$. To do this, we let $g \in L^2(\mathbb{R}^2)$ be such that $\hat{g} = f$. Then

$$(W_g e_j, e_k) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,\xi)\, W(e_j, e_k)(x,\xi)\, dx\, d\xi$$
$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(q,p)\, e_{j,k}(q,p)\, dq\, dp = 0$$

for $j,k = 0,1,2,\cdots$.

So, $W_g e_j = 0$, $j = 0, 1, 2, \cdots$. Now, let $h \in L^2(\mathbb{R})$

Let $\varepsilon > 0$ be any positive number. Since $\{e_j : j = 0, 1, 2, \cdots\}$ is an orthonormal basis for $L^2(\mathbb{R})$, we can find a nonnegative integer J such that

$$\| e_J - h \|_2 < \varepsilon.$$

So,

$$\| W_g h \|_2 = \| W_g (h - e_J) \|_2 + \| W_g e_J \|_2 \leq \varepsilon \| W_g \|_*,$$

where $\| W_g \|_*$ is the operator norm of W_g. Since ε and h are arbitrary, $W_g h = 0$, $h \in L^2(\mathbb{R})$. But then for all $h \in L^2(\mathbb{R})$,

$$(W_g h)(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(q, p) (\rho(q, p) h)(x) \, dq \, dp$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(q, p) e^{iq \cdot x + \frac{1}{2} iq \cdot p} h(x + p) \, dq \, dp = 0, \quad x \in \mathbb{R}.$$

Change the variable p to p' via $p' = x + p$. Then

$$(W_g h)(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} h(p') \underbrace{\left(\int_{-\infty}^{\infty} \hat{g}(q, p' - x) e^{iq \cdot x + \frac{1}{2} iq (p' - x)} dq \right) dp'}_{= 0}$$

for all $x \in \mathbb{R}$. Then

Then for almost all x and p' in $\mathbb{R}$,

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\hat{g}(q,p'-x)\,e^{iqx+\frac{1}{2}q(p'-x)}\,dq = 0.$$

So, by the Fourier inversion formula, we have for almost $x, p' \in \mathbb{R}$,

$$(\mathcal{F}_2 g)\left(\tfrac{1}{2}p' + \tfrac{1}{2}x,\ p'-x\right) = 0.$$

Let $q, \xi \in \mathbb{R}$. Then the linear equations

$$\begin{cases} \frac{1}{2}p' + \frac{1}{2}x = q \\ p' - x = \xi \end{cases}$$

has determinant

$$\det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} = -1 \neq 0.$$

So,

$$(\mathcal{F}_2 g)(q,\xi) = 0, \quad q,\xi \in \mathbb{R}.$$

$\therefore$ $g = 0$ almost everywhere on $\mathbb{R}^2$.