Lecture 14

For $k = 0, 1, 2, \cdots$, the Hermite function e_k of order k is the function e_k on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-\frac{x^2}{2}} H_k(x), \quad x \in \mathbb{R},$$

where $H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k \left(e^{-x^2}\right), \quad x \in \mathbb{R},$

is the Hermite polynomial of degree k.

Theorem: ① $\{ e_k : k = 0, 1, 2, \cdots \}$ is an orthonormal basis for $L^2(\mathbb{R})$.

② For $k = 0, 1, 2, \cdots$,

$$H e_k = (2k+1) e_k,$$

where $H = -\frac{d^2}{dx^2} + x^2$.

Remarks: ① means that for all $f \in L^2(\mathbb{R})$,

$$f = \sum_{k=0}^{\infty} (f, e_k) e_k,$$

where the convergence is in $L^2(\mathbb{R})$.

② means that for $k = 0, 1, 2, \cdots$, e_k is an eigenfunction of H and $2k+1$ is a corresponding eigenvalue.

Spectral Analysis of L_τ, $\tau \in \mathbb{R}\setminus\{0\}$

Let f and g be Schwartz functions on $\mathbb{R}$. Let $\tau \in \mathbb{R}\setminus\{0\}$. Then we define the τ-Fourier-Wigner transform $V_\tau(f,g)$ of f and g on $\mathbb{R}\times\mathbb{R}$ by

$$V_\tau(f,g)(q,p) = (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty} e^{i\tau q y} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}\,dy$$

for all $(q,p) \in \mathbb{R}\times\mathbb{R}$.

For $\tau \in \mathbb{R}\setminus\{0\}$, we define the function e_k^τ on $\mathbb{R}$, for $k = 0,1,2,\cdots$, by

$$e_k^\tau(x) = |\tau|^{1/4}\, e_k\left(\sqrt{|\tau|}\, x\right), \quad x \in \mathbb{R}.$$

For $j,k = 0,1,2,\cdots$, we define $e_{j,k}^\tau$ to be the function on $\mathbb{R}\times\mathbb{R}$ by

$$e_{j,k}^\tau = V_\tau(e_j^\tau, e_k^\tau).$$

If $\tau = 1$, then we denote $e_{j,k}^1$ by $e_{j,k}$, which is the ordinary Fourier-Wigner transform of the Hermite functions e_j and e_k.

<u>Question:</u> How is $e^{\tau}_{j,k}$, $\tau \in \mathbb{R}\setminus\{0\}$, related to $e_{j,k}$?

<u>Theorem:</u> Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for $j,k = 0,1,2,\dots$,

$$e^{\tau}_{j,k}(q,p) = |\tau|^{\frac{1}{2}} e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}}q, \sqrt{|\tau|}\,p\right)$$

for all $(q,p) \in \mathbb{R}\times\mathbb{R}$.

<u>Proof:</u>

$$\begin{aligned}
&e^{\tau}_{j,k}(q,p)\\
&= V_{\tau}(e^{\tau}_{j}, e^{\tau}_{k})(q,p)\\
&= (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty} e^{i\tau q y} e^{\tau}_{j}\left(y+\frac{p}{2}\right)\overline{e^{\tau}_{k}\left(y-\frac{p}{2}\right)}\,dy\\
&= (2\pi)^{-1/2}|\tau|\int_{-\infty}^{\infty} e^{i\tau q y} e_{j}\left(\sqrt{|\tau|}\left(y+\frac{p}{2}\right)\right)\overline{e_{k}\left(\sqrt{|\tau|}\left(y-\frac{p}{2}\right)\right)}\,dy.
\end{aligned}$$

Changing the variable from y to z via $z = \sqrt{|\tau|}\,y$, we get

$$\begin{aligned}
&e^{\tau}_{j,k}(q,p)\\
&= (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty} e^{i\tau q z/\sqrt{|\tau|}} e_{j}\left(z+\frac{\sqrt{|\tau|}\,p}{2}\right)\overline{e_{k}\left(z-\frac{\sqrt{|\tau|}\,p}{2}\right)}\,dz\\
&= |\tau|^{1/2} e_{j,k}\left(\frac{\tau q}{\sqrt{|\tau|}}, \sqrt{|\tau|}\,p\right), \quad (q,p)\in\mathbb{R}\times\mathbb{R}.
\end{aligned}$$