Lecture 13

Theorem: Let σ and τ be functions in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then $W_\sigma W_\tau = W_\omega$, where $\omega \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$

$$\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_{\frac{1}{4}} \hat{\tau}).$$

Proof: Let $z = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{C}^n$. Then for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,

$$(W_\sigma W_\tau f)(x)$$
$$= (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\sigma}(z)\big(\rho(z)(W_\tau f)\big)(x)\, dz.$$

But

$$(\rho(z)(W_\tau f))(x)$$
$$= e^{iq\cdot x + \frac{1}{2} iq\cdot p}(W_\tau f)(x+p)$$
$$= e^{iq\cdot x + \frac{1}{2} iq\cdot p}(2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\tau}(w)(\rho(w) f)(x+p)\, dw$$
$$= (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\tau}(w)\big(\rho(z)\rho(w) f\big)(x)\, dw$$
$$= (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\tau}(w)\big(\rho(z+w) e^{\frac{i}{4}[z,w]} f\big)(x)\, dw$$

$\therefore \ (W_\sigma W_\tau f)(x)$

$$= (2\pi)^{-2n} \int_{\mathbb{C}^n} \hat{\sigma}(z) \left[\int_{\mathbb{C}^n} \hat{\tau}(w) \left(\rho(z+w) e^{\frac{i}{4}[z,w]} f \right)(x)\, dw \right] dz.$$

Changing z to ζ via $\zeta = z+w$, using $[w,w] = 0$ and Fubini's theorem,

$(W_\sigma W_\tau f)(x)$

$$= (2\pi)^{-2n} \int_{\mathbb{C}^n} \hat{\sigma}(\zeta - w) \left[\int_{\mathbb{C}^n} \hat{\tau}(w) \left(\rho(\zeta) e^{\frac{i}{4}[\zeta - w, w]} f \right)(x)\, d\zeta \right] dw$$

$$= (2\pi)^{-2n} \left(\int_{\mathbb{C}^n} (\rho(\zeta) f)(x) \left\{ \int_{\mathbb{C}^n} \hat{\sigma}(\zeta - w) \hat{\tau}(w) e^{\frac{i}{4}[\zeta, w]}\, dw \right\} d\zeta \right)$$

$$= (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\omega}(\zeta) (\rho(\zeta) f)(x)\, d\zeta,$$

where

$$\hat{\omega}(\zeta) = (2\pi)^{-n} \int_{\mathbb{C}^n} \hat{\sigma}(\zeta - w) \hat{\tau}(w) e^{\frac{i}{4}[\zeta, w]}\, dw$$

$$= (2\pi)^{-n} (\hat{\sigma} *_{\frac{1}{4}} \hat{\tau}).$$

For $k=0,1,2,\cdots$, the Hermite function of order k is the function e_k on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k!\sqrt{\pi})^{1/2}} e^{-\frac{x^2}{2}} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k on $\mathbb{R}$ defined by

$$H_k(x) = (-1)^k e^{x^2/2} \left(\frac{d}{dx}\right)^k \left(e^{-x^2/2}\right), \quad x \in \mathbb{R}.$$

Let A and A^* be differential operators on $\mathbb{R}$ given by

$$A = \frac{d}{dx} + x$$

and

$$A^* = -\frac{d}{dx} + x.$$

The one-dimensional Hermite operator H on $\mathbb{R}$ is given by

$$H = \frac{1}{2}(AA^* + A^*A)$$

and a simple calculation gives

$$H = -\frac{d^2}{dx^2} + x^2.$$

<u>Theorem:</u> ① $\{e_k : k=0,1,2,\cdots\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

② $He_k = (2k+1)e_k, \; k=0,1,2,\cdots$.