Lecture 12

The Product of Two Weyl Transforms with Symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$

Lemma: Let $z, w \in \mathbb{C}^n$. Then $\rho(z)\rho(w) = \rho(z+w)e^{\frac{i}{4}[z,w]}$,

where

$$z = (z_1, z_2, \ldots, z_n),$$

$$w = (w_1, w_2, \ldots, w_n)$$

and

$$[z, w] = 2\,\mathrm{Im}\sum_{j=1}^{n} z_j \overline{w_j}.$$

Proof: Let $f \in L^2(\mathbb{R}^n)$ and let $x \in \mathbb{R}^n$. Write $z = q + ip$ and $w = v + iu$, where $q, p, u, v \in \mathbb{R}^n$. Then

$$(\rho(z)\rho(w)f)(x)$$
$$= e^{iq\cdot x + \frac{1}{2}iq\cdot p}(\rho(w)f)(x+p)$$
$$= e^{iq\cdot x + \frac{1}{2}iq\cdot p}\, e^{iv\cdot(x+p)}\, e^{\frac{1}{2}iv\cdot u} f(x+p+u)$$
$$= e^{iq\cdot x + \frac{1}{2}iq\cdot p + iv\cdot x + iv\cdot p + \frac{1}{2}iv\cdot u} f(x+p+u).$$

Next,

$$\left(\rho(z+w)e^{\frac{i}{4}[z,w]} f\right)(x)$$
$$= e^{\frac{i}{4}[z,w]}(\rho(z+w)f)(x).$$

$\therefore \left(\rho(z+w)e^{\frac{i}{4}[z,w]}f\right)(x)$

$= e^{\frac{i}{4}2(p\cdot v-q\cdot u)}\, e^{i(q+v)\cdot x+\frac{1}{2}i(q+v)\cdot(p+u)} f(x+p+u)$

$= e^{\frac{i}{2}p\cdot v-\cancel{\frac{i}{2}q\cdot u}+iq\cdot x+iv\cdot x+\frac{1}{2}iq\cdot p+\cancel{\frac{i}{2}q\cdot u}+\frac{1}{2}v\cdot p+\frac{i}{2}v\cdot u} f(x+p+u)$

$= e^{iv\cdot p+iq\cdot x+iv\cdot x+\frac{1}{2}iv\cdot u} f(x+p+u).$

$\therefore \rho(z)\rho(w)=\rho(z+w)e^{\frac{i}{4}[z,w]}, \quad z,w\in\mathbb{C}^n.$

Theorem: Let σ and τ be functions in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$, also known as symbols in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$. Then

$$W_\sigma W_\tau = W_\omega,$$

where

$$\hat{\omega}=(2\pi)^{-n}\left(\hat{\sigma} *_{\frac{1}{4}} \hat{\tau}\right).$$

Proof: Let $z=(q,p)\in\mathbb{R}^n\times\mathbb{R}^n\simeq\mathbb{C}^n$. Then for all $f\in L^2(\mathbb{R}^n)$ and all $x\in\mathbb{R}^n$,

$(W_\sigma W_\tau f)(x)$

$=(2\pi)^{-n}\int_{\mathbb{C}^n}\hat{\sigma}(z)\left(\rho(z)(W_\tau f)\right)(x)\,dz.$

But

$$(\rho(z)W_\tau f)(x)$$
$$= e^{iq\cdot x + \frac{1}{2}iq\cdot p}(W_\tau f)(x+p)$$
$$= e^{iq\cdot x+\frac{1}{2}iq\cdot p}(2\pi)^{-n}\int_{\mathbb{C}^n}\hat{\tau}(\omega)(\rho(\omega)f)(x+p)\,d\omega$$
$$=(2\pi)^{-n}\int_{\mathbb{C}^n}\hat{\tau}(\omega)(\rho(z)\rho(\omega)f)(x)\,d\omega$$
$$=(2\pi)^{-n}\int_{\mathbb{C}^n}\hat{\tau}(\omega)\left(\rho(z+\omega)e^{\frac{i}{4}[z,\omega]}f\right)(x)\,d\omega.$$

$\therefore$ $(W_\sigma W_\tau f)(x)$
$$=(2\pi)^{-2n}\int_{\mathbb{C}^n}\hat{\sigma}(z)\left[\int_{\mathbb{C}^n}\hat{\tau}(\omega)\left(\rho(z+\omega)e^{\frac{i}{4}[z,\omega]}f\right)(x)\,d\omega\right]dz.$$

Changing z to ζ via $z=\zeta-\omega$, using $[\omega,\omega]=0$ and Fubini's theorem,

$$(W_\sigma W_\tau f)(x)$$
$$=\left(\int_{\mathbb{C}^n}(\rho(\zeta)f)(x)\left(\int_{\mathbb{C}^n}\hat{\sigma}(\zeta-\omega)\hat{\tau}(\omega)e^{\frac{i}{4}[\zeta-\omega,\omega]}\,d\omega\right)d\zeta\right)(2\pi)^{-2n}$$
$$=(2\pi)^{-n}\int_{\mathbb{C}^n}\hat{\omega}(\zeta)(\rho(\zeta)f)(x)\,d\zeta,$$

where
$$\hat{\omega}(\zeta)=(2\pi)^{-n}\int_{\mathbb{C}^n}\hat{\sigma}(\zeta-\omega)\hat{\tau}(\omega)e^{\frac{i}{4}[\zeta,\omega]}\,d\omega$$
$$=(\hat{\sigma}*_{\frac{1}{4}}\hat{\tau})(2\pi)^{-n}.$$