Lecture 11

Theorem: (Moyal's Identity) Let f_1, g_1, f_2, g_2 be functions in $L^2(\mathbb{R}^n)$. Then

$$(W(f_1, g_1), W(f_2, g_2)) = (f_1, f_2)\overline{(g_1, g_2)}.$$

Proof:

$$(W(f_1, g_1), W(f_2, g_2))$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W(f_1, g_1)(x,\xi)\,\overline{W(f_2, g_2)(x,\xi)}\,dx\,d\xi.$$

But for $j = 1, 2$,

$$W(f_j, g_j)(x,\xi)$$
$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p} f_j\left(x+\frac{p}{2}\right)\overline{g_j\left(x-\frac{p}{2}\right)}\,dp.$$

By Plancherel's theorem,

$$(W(f_1, g_1), W(f_2, g_2))$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1\left(x+\frac{p}{2}\right) g_1\left(x-\frac{p}{2}\right)\overline{f_2\left(x+\frac{p}{2}\right)} g_2\left(x-\frac{p}{2}\right)dx\,dp.$$

Let $u = x + \frac{p}{2}$, $v = x - \frac{p}{2}$. Then

$$du\,dv = \left|\det\begin{pmatrix} I & \frac{1}{2}I \\ I & -\frac{1}{2}I \end{pmatrix}\right| dx\,dp = dx\,dp.$$

$\therefore \ (W(f_1, g_1), W(f_2, g_2))$

$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1(u) \overline{g_1(v)} \, \overline{f_2(u)} \, g_2(v) \, du dv$

$= (f_1, f_2) \overline{(g_1, g_2)}.$

<u>Weyl Transforms</u>: Let σ be a function in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then for all functions f in $L^2(\mathbb{R}^n)$, we define the Weyl transform of f with symbol σ by

$$(W_\sigma f, g) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi) \, dx d\xi$$

for all g in $L^2(\mathbb{R}^n)$.

<u>Another Formula for the Weyl transforms</u>:

By the adjoint formula for the Fourier transform,

$(W_\sigma f, g) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) \, V(f, g)^\wedge(x, \xi) \, dx d\xi$

$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q, p) \, V(f, g)(q, p) \, dq dp$

$\therefore (W_\sigma f, g)$

$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) (\rho(q,p) f, g) \, dq \, dp.$

$\therefore W_\sigma f = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) \rho(q,p) f \, dq \, dp.$

Products of Weyl Transforms: We need a lemma.

Lemma: Let z and w be points in $\mathbb{C}^n$. Then

$$\rho(z)\rho(w) = \rho(z+w) e^{\frac{1}{4} i [z, w]}, \text{ where}$$

$$[z, w] = 2 \, \mathrm{Im}(z \cdot \overline{w})$$

and $z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w_j}$.

Proof: Let $z = q + ip$, $w = v + iu$. Then for all f in $L^2(\mathbb{R}^n)$,

$(\rho(z)\rho(w) f)(x)$

$= e^{iq\cdot x + \frac{1}{2} i q \cdot p} (\rho(w) f)(x+p)$

$= e^{iq\cdot x + \frac{1}{2} i q \cdot p} e^{iv\cdot(x+p) + \frac{1}{2} i v \cdot u} f(x+p+u)$

$= e^{iq\cdot x + \frac{1}{2} i q \cdot p + iv \cdot x + iv \cdot p + \frac{1}{2} i v \cdot u} f(x+p+u),$

$x \in \mathbb{R}^n.$