Lecture 1

The Heisenberg Group: We identify points in $\mathbb{R}^2$ via

$$\mathbb{R}^2 \ni (x,y) \leftrightarrow z = x+iy \in \mathbb{C}.$$

Let $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$. Then for all (z,t) and (w,s) in $\mathbb{H}^1$, we define $(z,t)\cdot(w,s)$ by

$$(z,t)\cdot(w,s) = \left(z+w,\, t+s+\tfrac{1}{4}[z,w]\right),$$

where $[z,w]$ is the symplectic form of z and w given by

$$[z,w] = 2\,\mathrm{Im}(z\bar{w}).$$

1. $\mathbb{H}^1$ is a noncommutative group with respect to $\cdot$.
2. The identity element is $(0,0)$.
3. The inverse $(z,t)^{-1}$ of every element (z,t) in $\mathbb{H}^1$ is $(-z,-t)$

Proposition: The Lebsgue measure $dz\,dt = dx\,dy\,dt$ on $\mathbb{H}^1$ has the properties that for all measurable functions f on $\mathbb{H}^1$,

- $\displaystyle\int_{-\infty}^{\infty}\int_{\mathbb{C}} f((w,s)\cdot(z,t))\,dz\,dt = \int_{-\infty}^{\infty}\int_{\mathbb{C}} f(z,t)\,dz\,dt$ (Left Invariance)

and

- $\displaystyle\int_{-\infty}^{\infty}\int_{\mathbb{C}} f((z,t)\cdot(w,s))\,dz\,dt = \int_{-\infty}^{\infty}\int_{\mathbb{C}} f(z,t)\,dz\,dt$ (Right Invariance) for all (ω, s) in $\mathbb{H}'$.

Proof: (Left Invariance Only) Let $\omega = (u, v)$ and $z = (x, y)$. Then

$$[\omega, z] = \operatorname{Im}(\omega \bar{z}) = \operatorname{Im}((u+iv)(x-iy)) = vx - uy.$$

$$\therefore \int_{-\infty}^{\infty} \int_{\mathbb{C}} f((\omega, s)\cdot(z, t))\, dz\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(u+x, v+y, s+t+\tfrac{1}{2}(vx-uy)\right) dx\, dy\, dt.$$

Now, we make a change of variables from (x, y, t) to (ξ, η, τ) by $\xi = u+x$, $\eta = v+y$, $\tau = s+t+\frac{1}{2}(vx-uy)$.

Then $d\xi\, d\eta\, d\tau = \left| \det J\begin{pmatrix} \xi & \eta & \tau \\ x & y & t \end{pmatrix} \right| dx\, dy\, dt$, where

$$J\begin{pmatrix} \xi & \eta & \tau \\ x & y & t \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial t} \\ \frac{\partial \tau}{\partial x} & \frac{\partial \tau}{\partial y} & \frac{\partial \tau}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}v & -\frac{1}{2}u & 1 \end{pmatrix}.$$

Therefore $d\xi\, d\eta\, d\tau = dx\, dy\, dt$. Let $\zeta = (\xi, \eta)$. Then

$$\int_{-\infty}^{\infty} \int_{\mathbb{C}} f((\omega, s)\cdot(z, t))\, dz\, dt = \int_{-\infty}^{\infty} \int_{\mathbb{C}} f(\zeta, \tau)\, d\zeta\, d\tau.$$

Remarks: 1. The Heisenberg group is a Lie group.

2. On any Lie group G, there is a unique measure on G that is invariant with respect to the left translation induced by the group law on G. Uniqueness means up to a positive scalar multiple. It is called a left Haar measure on G. Similar remarks hold for right Haar measures.

3. A Lie group on which the left Haar measure and the right Haar measure are the same is said to be unimodular. So, the Heisenberg group $\mathbb{H}^1$ is unimodular.

A Lie algebra $\mathfrak{g}$ is a real vector space on which there is a binary operation $[\,,\,]$ such that $[\,,\,]$ is bilinear and the Jacobi identity

$$[g_1,[g_2,g_3]] + [g_2,[g_3,g_1]] + [g_3,[g_1,g_2]] = 0$$

We call $[\,,\,]$ the bracket in $\mathfrak{g}$.

To give an example of a Lie algebra, we look at vector fields on $\mathbb{H}^1$.

<u>Definition</u>: A vector field V on $\mathbb{H}^1$ given by

$$V(x,y,t) = a(x,y,t)\frac{\partial}{\partial x} + b(x,y,t)\frac{\partial}{\partial y} + c(x,y,t)\frac{\partial}{\partial t}$$

for all $(x,y,t) \in \mathbb{H}^1$, where a, b, c are real-valued C^∞ functions on $\mathbb{H}^1$, is said to be <u>left-invariant</u> if

$$V L_{(\omega,s)} = L_{(\omega,s)} V$$

for all $(\omega,s) \in \mathbb{H}^1$, where $L_{(\omega,s)}$ is the left translation by (ω,s) given by

$$(L_{(\omega,s)} f)(z,t) = f((\omega,s)\cdot(z,t)), \quad (z,t) \in \mathbb{H}^1.$$

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