Lecture 7

Definition: Let $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$, $x \in \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$. Then we say that $P(x,D)$ is elliptic at x_0 if

$$P_m(x_0,\xi) = 0,\ \xi \in \mathbb{R}^n \Rightarrow \xi = 0.$$

If $P(x,D)$ is elliptic at every point in $\mathbb{R}^n$, then we say that $P(x,D)$ is elliptic on $\mathbb{R}^n$.

A Well-Known Fact: Suppose that $P(x,D)$ is elliptic on $\mathbb{R}^n$. Then

$$u \in \mathcal{D}'(\mathbb{R}^n),\ P(x,D)u \in C^\infty(\mathbb{R}^n) \Rightarrow u \in C^\infty(\mathbb{R}^n),$$

where $\mathcal{D}'(\mathbb{R}^n)$ is the space of all distributions on $\mathbb{R}^n$.

Remarks: 1. Let Ω be an open and connected subset of $\mathbb{R}^n$. Let $C_0^\infty(\Omega)$ be the set of all $\varphi \in C^\infty(\Omega)$ such that $\mathrm{supp}(\varphi)$ is a compact subset of Ω.

2. Convergence in $C_0^\infty(\Omega)$: Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence in $C_0^\infty(\Omega)$. Suppose that there exists a compact subset K of Ω for which

$$\mathrm{supp}(\varphi_j) \subseteq K,\ j = 1, 2, \cdots,$$

and for all multi-indices α,

$$D^\alpha \varphi_j \to 0$$

uniformly on Ω as $j \to \infty$. Then we say that $\varphi_j \to 0$ in $C_0^\infty(\Omega)$ as $j \to \infty$.

3. Let $T : C_0^\infty(\Omega) \to \mathbb{C}$ be a linear functional such that for all sequences $\{\varphi_j\}_{j=1}^\infty$ in $C_0^\infty(\Omega)$ with $\varphi_j \to 0$ in $C_0^\infty(\Omega)$, $T(\varphi_j) \to 0$ as $j \to \infty$. Then we call T a <u>distribution</u> on Ω.

4. We denote the space of all distributions on Ω by $\mathcal{D}'(\Omega)$.

5. $\mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$. (Exercise)

6. We say that $P(x,D)$ is <u>hypoelliptic</u> on Ω if

$$u \in \mathcal{D}'(\Omega),\ P(x,D)u \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega).$$

<u>Theorem</u>: Let $\tau \in \mathbb{R} \setminus \{0\}$. Then L_τ is elliptic on $\mathbb{R}^2$.

<u>Proof</u>: $L_\tau = -\Delta + \frac{1}{4}(x^2+y^2)\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau$.

The principal part is $-\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. So,

$$-\Delta = +\left(\left(-i\frac{\partial}{\partial x}\right)^2 + \left(-i\frac{\partial}{\partial y}\right)^2\right).$$

$\therefore$ the principal symbol of $-\Delta$ is $\xi_1^2 + \xi_2^2 = |\xi|^2$, $\xi \in \mathbb{R}^2$.

$$\therefore\ |\xi|^2 = 0 \Rightarrow \xi = 0,\ \xi \in \mathbb{R}^2.$$

$\therefore L_\tau$ is elliptic on $\mathbb{R}^2$.

Theorem: $\mathcal{L}$ is nowhere elliptic on $\mathbb{R}^3$.

Proof: $\mathcal{L} = -(X^2+Y^2)$

$$= -\Delta - \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}$$

$$= -\left(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t}\right)^2.$$

To find its symbol $P(x,y,t;\xi,\eta,\tau)$, recall

$$-i\frac{\partial}{\partial x} \to \xi,\ -i\frac{\partial}{\partial y} \to \eta,\ -i\frac{\partial}{\partial t} \to \tau$$

or

$$\frac{\partial}{\partial x} \to i\xi,\ \frac{\partial}{\partial y} = i\eta,\ \frac{\partial}{\partial t} = i\tau.$$

So, the symbol, also the principal symbol, of $\mathcal{L}$ is

$$P(x,y,t;\xi,\eta,\tau) = -\left(i\xi + \frac{1}{2}yi\tau\right)^2 - \left(i\eta - \frac{1}{2}xi\tau\right)^2$$

$$= \left(\xi + \frac{1}{2}y\tau\right)^2 + \left(\eta - \frac{1}{2}x\tau\right)^2.$$

Let $(x_0,y_0,t_0) \in \mathbb{R}^3$. If $(x_0,y_0) = (0,0)$, then

$P(x_0,y_0,t_0;\xi,\eta,\tau) = 0 \Rightarrow \xi = \eta = 0$. So, τ can be anything. So $P(x_0,y_0,t_0;\xi,\eta,\tau) = 0$ for all $(0,0,\tau) \in \mathbb{R}^3$.

Now, suppose that $x_0 \neq 0$ or $y_0 \neq 0$. (Assume $x_0 \neq 0$.)

Then

$$P(x_0, y_0, t_0; \xi, \eta, \tau) = 0$$

for all (ξ, η, τ) with

$$\begin{cases} \xi = -\frac{1}{2} y_0 \tau, \\ \eta = \frac{1}{2} x_0 \tau. \end{cases}$$

$\therefore \mathcal{L}$ is nowhere elliptic on $\mathbb{R}^3$.

Remark: By a result of Hörmander in 1967 in the context of Heisenberg groups, if all the "horizontal" vector fields and all their commutators up to a certain order span the corresponding Lie algebra, then the sum of the squares of the horizontal vector fields is hypoelliptic.