Lecture 10

We identify points in $\mathbb{R}^n \times \mathbb{R}^n$ with points in $\mathbb{C}^n$ via the identification

$$\mathbb{R}^n \times \mathbb{R}^n \ni (q,p) \leftrightarrow q + ip \in \mathbb{C}^n.$$

Let q and p be points in $\mathbb{R}^n$. Then we define the function $\rho(q,p)f$ on $\mathbb{R}^n$ by

$$(\rho(q,p)f)(x) = e^{iq\cdot x + \frac{1}{2}iq\cdot p} f(x+p), \quad x \in \mathbb{R}^n.$$

Let f and g be Schwartz functions on $\mathbb{R}^n$. Then we define the Fourier-Wigner transform $V(f,g)$ of f and g to be the function on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$V(f,g)(q,p) = (2\pi)^{-n/2} (\rho(q,p)f, g), \quad q,p \in \mathbb{R}^n.$$

<u>Theorem</u>: The Fourier-Wigner transform $V(f,g)$ of the Schwartz functions f and g on $\mathbb{R}^n$ is given by

$$V(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)}\, dy$$

for all $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$.

<u>Proof</u>: For all $q,p \in \mathbb{R}^n$,

$$V(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq\cdot x + \frac{1}{2}q\cdot p} f\left(x+\frac{p}{2}\right) \overline{g\left(x-\frac{p}{2}\right)}\, dx.$$

Let $x = y - \frac{p}{2}$. Then

$$V(f,g)(q,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iqy} f\left(y+\frac{p}{2}\right)\overline{g\left(y+\frac{p}{2}\right)}\,dy.$$

Let f and g be Schwartz functions on $\mathbb{R}^n$. Then the Wigner transform $W(f,g)$ of f and g is defined by

$$W(f,g) = V(f,g)^{\wedge}.$$

Proof: We have for all $x, \xi \in \mathbb{R}^n$,

$$W(f,g)(x,\xi)$$

$$= (2\pi)^{-3n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot q - i\xi\cdot p}\left(\int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}\,dy\right)dq\,dp$$

$$= (2\pi)^{-3n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(e^{-iq\cdot(x-y)}\,dq\right)e^{-i\xi\cdot p} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}\,dy\,dp.$$

Since

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iq(y-x)}\,dq = \delta(y-x) = \delta(x-y),$$

$x, y \in \mathbb{R}^n$, $\therefore$

$$W(f,g)(x,\xi)$$

$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p}\left(\int_{\mathbb{R}^n}\delta(x-y) f\left(y+\frac{p}{2}\right)\overline{g\left(y+\frac{p}{2}\right)}\,dy\right)dp$$

$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n}^{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}\,dp.$$

Proposition: Let $f, g \in L^2(\mathbb{R}^n)$. Then
$$W(g,f) = \overline{W(f,g)}. \quad (\text{Exercise})$$

Theorem: (Moyal's Identity) Let $f_1, g_1, f_2, g_2 \in L^2(\mathbb{R}^n)$.
Then $(W(f_1,g_1), W(f_2,g_2)) = (f_1, f_2)\overline{(g_1,g_2)}$.

Proof: By Plancherel's theorem,
$$(W(f_1,g_1), W(f_2,g_2)) = (V(f_1,g_1)^\wedge, V(f_2,g_2)^\wedge)$$
$$= (V(f_1,g_1), V(f_2,g_2)) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1\left(x+\frac{p}{2}\right)\overline{g_1\left(x-\frac{p}{2}\right)}\,\overline{f_2\left(x+\frac{p}{2}\right)}\,\overline{g_2\left(x-\frac{p}{2}\right)}\,dx\,dp.$$

Let $u = x + \frac{p}{2}$, $v = x - \frac{p}{2}$. Then
$$du\,dv = \left|\det\begin{pmatrix} I & \frac{1}{2}I \\ I & -\frac{1}{2}I \end{pmatrix}\right| dx\,dp = dx\,dp.$$

So
$$(W(f_1,g_1), W(f_2,g_2))$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(u)\overline{g_1(v)}\,\overline{f_2(u)}\,g_2(v)\,du\,dv$$
$$= (f_1, f_2)\overline{(g_1, g_2)}.$$