C^*-Algebras, H^*-Algebras and Trace Ideals of Pseudo-Differential Operators on Locally Compact, Hausdorff and Abelian Groups

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Abstract We define pseudo-differential operators on a locally compact, Hausdorff and abelian group G as natural extensions of pseudo-differential operators on $\mathbb{R}^n$. In particular, for pseudo-differential operators with symbols in $L^2(G \times \widehat{G})$, where $\widehat{G}$ is the dual group of G, we give explicit formulas for the products and adjoints, characterize them as Hilbert–Schmidt operators on $L^2(G)$ and prove that they form a C^*-algebra, which is also a H^*-algebra. We give a characterization of trace class pseudo-differential operators in terms of symbols lying in a subspace of $L^1(G\times\widehat{G})\cap L^2(G\times\widehat{G})$.

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1 Introduction

Throughout this paper, G is a locally compact, Hausdorff and abelian group with binary operation $\cdot$ and Haar measure μ. Let ξ be a continuous complex-valued function on G such that

$$|\xi(x)| = 1, \quad x \in G,$$

and

$$\xi(x \cdot y) = \xi(x)\xi(y), \quad x, y \in G.$$

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Then ξ is known as a *character* of G. We denote by $\widehat{G}$ the set of all characters of G. Then $\widehat{G}$ is a group with respect to the usual multiplication of functions. In fact, $\widehat{G}$ is also a locally compact, Hausdorff and abelian group. The group $\widehat{G}$ is known as the *dual group* of G. The Haar measure on $\widehat{G}$, to be suitably normalized in Theorem 1.1, is denoted by ν.

Let $u \in L^1(G)$. Then the *Fourier transform* of u is the function $\mathcal{F}_G u$ on $\widehat{G}$ defined by

$$(\mathcal{F}_G u)(\xi) = \int_G \overline{\xi(x)} u(x)\, d\mu(x), \quad \xi \in \widehat{G}.$$

We have the following theorem, which is the *Plancherel theorem* for G.

Theorem 1.1 *The Fourier transform, initially defined on $L^1(G) \cap L^2(G)$, can be extended uniquely to a surjective isometry $\mathcal{F}_G : L^2(G) \to L^2(\widehat{G})$, which is called the Fourier transform on $L^2(G)$. We can normalize the Haar measure ν on $\widehat{G}$ in such a way that*

$$(\mathcal{F}_G u, \mathcal{F}_G v)_{L^2(\widehat{G})} = (u, v)_{L^2(G)}$$

for all $u, v \in L^2(G)$.

The following *Fourier inversion formula* for Fourier transforms on G is the result for building a theory of pseudo-differential operators on G.

Theorem 1.2 *For all $u \in L^2(G)$, we have*

$$u = \int_{\widehat{G}} \xi(\cdot)(\mathcal{F}_G u)(\xi)\, d\nu(\xi).$$

Remark 1.3 The Fourier inversion formula can be reformulated as

$$u = \mathcal{F}_G^{-1} \mathcal{F}_G u, \quad u \in L^2(G),$$

where $\mathcal{F}_G^{-1} : L^2(\widehat{G}) \to L^2(G)$ is given by

$$(\mathcal{F}_G^{-1}\varphi)(x) = \int_{\widehat{G}} \xi(x)\varphi(\xi)\, d\nu(\xi), \quad x \in G,$$

for all $\varphi \in L^2(\widehat{G})$.

In this paper we assume that the Haar measure on $\widehat{G}$ is so normalized that the Plancherel theorem is valid. The Fourier analysis hitherto described can be found in [4, 5, 12]. See also [15] for Haar measures and integral operators on locally compact and Hausdorff groups.

We can now give the definition of pseudo-differential operators on G. Let σ be a function on the phase space $G \times \widehat{G}$. Then we define the *pseudo-differential operator* T_σ on G corresponding to the symbol σ by

$$(T_\sigma u)(x) = \int_{\widehat{G}} \xi(x)\sigma(x,\xi)(\mathcal{F}_G u)(\xi)\, d\nu(\xi), \quad x \in G,$$

for all suitable functions u on G such that the integral exists.

Remark 1.4 The Fourier inversion formula gives another expression for the identity operator on $L^2(G)$. The identity operator on $L^2(G)$ can be thought of as a *perfect symmetry* and the role of a symbol is to *break the symmetry* in order to get a pseudo-differential operator on $L^2(G)$, i.e., a *broken symmetry*.

Pseudo-differential operators on $\mathbb{R}^n$, the unit circle $\mathbb{S}^1$ with center at the origin, the additive group $\mathbb{Z}$ of all integers and finite abelian groups studied in [7, 8, 9, 10, 13, 17, 18] are all special cases of pseudo-differential operators on locally compact, Hausdorff and abelian groups.

In Section 2, we show that a pseudo-differential operator with symbol σ in $L^2(G\times\widehat{G})$ is a bounded linear operator from $L^2(G)$ into $L^2(G)$. We then prove in Section 3 that $T_\sigma : L^2(G) \to L^2(G)$ is a Hilbert–Schmidt operator if and only if $\sigma \in L^2(G \times \widehat{G})$. Explicit formulas for the products and adjoints of pseudo-differential operators are given in Section 4. A characterization of trace class pseudo-differential operators on $L^2(G)$ is given in Section 5. We use the products and the adjoints in Section 4 to show that pseudo-differential operators with symbols in $L^2(G \times \widehat{G})$ form a C^*-algebra that is at the same time a H^*-algebra. An immediate by-product is that $L^2(G\times\widehat{G})$ can be turned into a C^*-algebra, which is also a H^*-algebra, when equipped with a suitable product and a suitable involution.

2 Bounded Linear Operators

We first give a sufficient condition on σ to guarantee that the pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is a bounded linear operator.

Theorem 2.1 *Let $\sigma \in L^2(G \times \widehat{G})$. Then the pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is a bounded linear operator. Moreover,*

$$\|T_\sigma\|_{B(L^2(G))} \leq \|\sigma\|_{L^2(G\times\widehat{G})},$$

where $\| \, \|_{B(L^2(G))}$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(G)$.

Proof Let $u \in L^2(G)$. Then by Minkowski's inequality in integral form,

$$\begin{aligned}
\|T_\sigma u\|_{L^2(G)} &= \left(\int_G |(T_\sigma u)(x)|^2 d\mu(x) \right)^{1/2} \\
&= \left(\int_G \left| \int_{\widehat{G}} \xi(x)\sigma(x,\xi)(\mathcal{F}_G u)(\xi)\, d\nu(\xi) \right|^2 d\mu(x) \right)^{1/2} \\
&\leq \int_{\widehat{G}} \left(\int_G |\sigma(x,\xi)|^2 |(\mathcal{F}_G u)(\xi)|^2 d\mu(x) \right)^{1/2} d\nu(\xi) \\
&= \int_{\widehat{G}} |(\mathcal{F}_G u)(\xi)| \left(\int_G |\sigma(x,\xi)|^2 d\mu(x) \right)^{1/2} d\nu(\xi).
\end{aligned}$$

Using first the Schwarz inequality and then the Plancherel formula, we get

$$\begin{aligned}
\|T_\sigma u\|_{L^2(G)} &\leq \left(\int_{\widehat{G}} \int_G |\sigma(x,\xi)|^2 d\mu(x)\, d\nu(\xi) \right)^{1/2} \left(\int_{\widehat{G}} |(\mathcal{F}_G u)(\xi)|^2 d\nu(\xi) \right)^{1/2} \\
&= \left(\int_{\widehat{G}} \int_G |\sigma(x,\xi)|^2 d\mu(x)\, d\nu(\xi) \right)^{1/2} \|\mathcal{F}_G u\|_{L^2(\widehat{G})} \\
&= \left(\int_{\widehat{G}} \int_G |\sigma(x,\xi)|^2 d\mu(x)\, d\nu(\xi) \right)^{1/2} \|u\|_{L^2(G)}
\end{aligned}$$

and the proof is complete. □

3 Hilbert–Schmidt Operators

We can improve Theorem 2.1 to the following theorem.

Theorem 3.1 *The pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ with symbol σ is a Hilbert–Schmidt operator if and only if $\sigma \in L^2(G \times \widehat{G})$.*

Moreover, for all $\sigma \in L^2(G \times \widehat{G})$, the Hilbert–Schimdt norm $\|T_\sigma\|_{S_2(G)}$ of $T_\sigma : L^2(G) \to L^2(G)$ is given by

$$\|T_\sigma\|_{S_2(G)} = \|\sigma\|_{L^2(G\times\widehat{G})}.$$

Proof Let $\sigma \in L^2(G)$. Then for all functions $u \in L^2(G)$, we have

$$\begin{aligned}
(T_\sigma u)(x) &= \int_{\widehat{G}} \xi(x)\sigma(x,\xi)(\mathcal{F}_G u)(\xi)\, d\nu(\xi) \\
&= \int_{\widehat{G}} \xi(x)\sigma(x,\xi) \left(\int_G \overline{\xi(y)}\, u(y)\, d\mu(y) \right) d\nu(\xi) \\
&= \int_G \left(\int_{\widehat{G}} \xi(x)\overline{\xi(y)}\sigma(x,\xi)\, d\nu(\xi) \right) u(y)\, d\mu(y) \\
&= \int_G K(x,y)u(y)\, d\mu(y),
\end{aligned}$$

where

$$K(x,y) = \int_{\widehat{G}} \xi(x)\overline{\xi(y)}\sigma(x,\xi)\, d\nu(\xi), \quad x, y \in G.$$

For almost all x and y in G,

$$K(x,y) = \int_{\widehat{G}} \xi(x\cdot y^{-1})\sigma(x,\xi)\, d\nu(\xi) = (\mathcal{F}_G^{-1}\sigma)(x, x\cdot y^{-1}). \tag{3.1}$$

Thus,

$$\begin{aligned}
\|K\|^2_{L^2(G\times G)} &= \int_G\int_G |K(x,y)|^2 d\mu(x)\, d\mu(y) \\
&= \int_G\int_G |(\mathcal{F}_G^{-1}\sigma)(x, x\cdot y^{-1})|^2 d\mu(x)\, d\mu(y).
\end{aligned}$$

If we let $z = x\cdot y^{-1}$, then using Plancherel's formula, we have

$$\begin{aligned}
\|K\|^2_{L^2(G\times G)} &= \int_G\int_G |(\mathcal{F}_G^{-1}\sigma)(x, x\cdot y^{-1})|^2 d\mu(y)\, d\mu(x) \\
&= \int_G\int_G |(\mathcal{F} - G^{-1}\sigma)(x,z)|^2 d\mu(z)\, d\mu(y) \\
&= \int_G\int_{\widehat{G}} |\sigma(x,\xi)|^2 d\nu(\xi)\, d\mu(x) \\
&= \|\sigma\|^2_{L^2(G\times G)}.
\end{aligned}$$

Therefore $T_\sigma : L^2(G) \to L^2(G)$ is a Hilbert–Schmidt operator. Conversely, let σ be a symbol defined on $G \times \widehat{G}$ such that the corresponding pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is a Hilbert–Schimdt operator. Then there exists a function $K \in L^2(G \times G)$ such that for all $u \in L^2(G)$,

$$(T_\sigma u)(x) = \int_G K(x,y)\, d\mu(y), \quad x \in G.$$

By Plancherel's formula,

$$(T_\sigma u)(x) = \int_G \overline{\overline{K}(x,y)} u(y)\, d\mu(y) = \int_G \overline{(\mathcal{F}_{G,2}\overline{K})(x,\xi)} (\mathcal{F}_G u)(\xi)\, d\nu(\xi)$$

for almost all $x \in G$, where $\mathcal{F}_{G,2}\overline{K}$ is the Fourier transform of $\overline{K}$ with respect to the second varaiable. But for almost all $\xi \in \widehat{G}$,

$$(\mathcal{F}_{G,2}\overline{K})(x,\xi) = \int_G \overline{\xi(y) K(x,y)} d\mu(y).$$

So, for almost all $x \in G$,

$$\begin{aligned}
(T_\sigma u)(x) &= \int_G \left(\int_G \xi(y) K(x,y)\, d\mu(y) \right) (\mathcal{F}_G u)(\xi)\, d\nu(\xi) \\
&= \int_G \xi(x) \left(\int_G \overline{\xi(x)} \xi(y) K(x,y)\, d\mu(y) \right) (\mathcal{F}_G u)(\xi)\, d\nu(\xi).
\end{aligned}$$

Hence for almost all $(x,\xi) \in G \times \widehat{G}$,

$$\sigma(x,\xi) = \overline{\xi(x)} (\mathcal{F}_{G,2}K)(x,\xi^{-1}), \tag{3.2}$$

where $\mathcal{F}_{G,2}K$ denotes the Fourier transform of K with respect to the second variable, and we have by means of Plancherel's formula,

$$\begin{aligned}
&\int_G \int_{\widehat{G}} |\sigma(x,\xi)|^2 d\nu(\xi)\, d\mu(x) \\
= &\int_G \int_{\widehat{G}} |(\mathcal{F}_{G,2}K)(x,\xi)|^2 d\nu(\xi)\, d\mu(x) \\
= &\int_G \int_G |K(x,y)|^2 d\mu(y)\, d\mu(x) \\
< &\ \infty.
\end{aligned}$$

Therefore $\sigma \in L^2(G \times \widehat{G})$. Finally, let $\sigma \in L^2(G \times \widehat{G})$. Then T_σ is a Hilbert–Schmidt operator on $L^2(G)$ and its Hilbert–Schmidt norm $\|T_\sigma\|_{S_2(G)}$ is given by

$$\|T_\sigma\|_{S_2(G)} = \left(\int_G \int_{\widehat{G}} |\sigma(x,\xi)|^2 d\nu(\xi)\, d\mu(x)\right)^{1/2} = \|\sigma\|_{L^2(G\times\widehat{G})},$$

as asserted. □

By Theorems 2.1 and 3.1, we have the following corollary.

Corollary 3.2 *Let $\sigma \in L^2(G \times \widehat{G})$. Then*

$$\|T_\sigma\|_{B(L^2(G))} \leq \|\sigma\|_{L^2(G\times\widehat{G})} = \|T_\sigma\|_{S_2(G)}.$$

4 Products and Adjoints

We begin this section with a formula for the product of two pseudo-differential operators on $L^2(G)$.

Theorem 4.1 *Let σ and τ be functions on $L^2(G \times \widehat{G})$. Then $T_\sigma T_\tau = T_\lambda$, where $\lambda \in L^2(G \times \widehat{G})$ is given by*

$$\lambda(x,\xi) = \int_G \xi(z)(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})\tau(z,\xi^{-1})\, d\mu(z), \quad (x,\xi) \in G \times \widehat{G}.$$

Moreover,

$$\|\lambda\|_{L^2(G\times\widehat{G})} \leq \|\sigma\|_{L^2(G\times\widehat{G})}\|\tau\|_{L^2(G\times\widehat{G})}.$$

Proof Let $u \in L^2(G)$. Then we get by (3.1)

$$(T_\sigma u)(x) = \int_G K_\sigma(x,y)u(y)\, d\mu(y), \quad x \in \mathbb{R}^n,$$

where

$$K_\sigma(x,y) = (\mathcal{F}_G^{-1}\sigma)(x, x\cdot y^{-1}), \quad x,y \in G,$$

and

$$(T_\tau u)(x) = \int_G K_\tau(x,y)u(y)\, d\mu(y), \quad x \in G,$$

where

$$K_\tau(x,y) = (\mathcal{F}_G^{-1}\tau)(x, x\cdot y^{-1}), \quad x,y \in G.$$

Therefore

$$(T_\sigma T_\tau u)(x) = \int_G K(x,y)u(y)\,d\mu(y), \quad x \in \mathbb{R}^n,$$

where

$$K(x,y) = \int_G (\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})H(z,y)\,dz, \quad x,y \in G,$$

where

$$H(z,y) = (\mathcal{F}_G^{-1}\tau)(z, z\cdot y^{-1}), \quad z,y \in G.$$

By (3.2), the symbol λ of $T_\sigma T_\tau$ is then given by

$$\lambda(x,\xi) = \overline{\xi(x)} \int_G (\mathcal{F}_G^{-1}\sigma)(x, z^{-1})(\mathcal{F}_{G,2}H)(z,\xi^{-1})\,d\mu(z),$$

where $\mathcal{F}_{G,2}H$ is the Fourier transform of H on G with respect to the second variable given by

$$\begin{aligned}
(\mathcal{F}_{G,2}H)(z,\xi^{-1}) &= \int_G \overline{\xi^{-1}(y)}(\mathcal{F}_G^{-1}\tau)(z, z\cdot y^{-1})\,d\mu(y) \\
&= \int_G \overline{\xi^{-1}(w^{-1}z)}(\mathcal{F}_G^{-1}(z,w)\,d(w^{-1}) \\
&= \overline{\xi^{-1}(z)} \int_G \overline{\xi^{-1}(w)}(\mathcal{F}_G^{-1}\tau(z,w)\,dw \\
&= \xi(z)(\mathcal{F}_G\mathcal{F}_G^{-1}\tau)(z,\xi^{-1}) \\
&= \xi(z)\tau(z,\xi^{-1}), \quad (z,\xi) \in G\times\widehat{G}.
\end{aligned}$$

Therefore

$$\lambda(x,\xi) = \int_G \xi(z)(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})\tau(z,\xi^{-1})\,d\mu(z), \quad (x,\xi) \in G\times\widehat{G}.$$

Now, by Minkowski's inequality in integral form and Fubini's theorem,

$$\begin{aligned}
&\|\lambda\|_{L^2(G\times\widehat{G})} \\
= &\left(\int_{\widehat{G}}\int_G |\lambda(x,\xi)|^2 d\mu(x)\,d\nu(\xi)\right)^{1/2} \\
= &\left(\int_{\widehat{G}}\int_G \left|\int_G \xi(z)(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})\tau(z,\xi^{-1})\,d\mu(z)\right|^2 d\mu(x)\,d\nu(\xi)\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le \int_G \left(\int_{\widehat{G}}\int_G |(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})|^2 |\tau(z,\xi^{-1})|^2 d\mu(x)\, d\nu(\xi)\right)^{1/2} d\mu(z) \\
&= \int_G \left[\int_{\widehat{G}} |\tau(z,\xi^{-1})|^2 \left(\int_G |(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})|^2 d\mu(x)\right) d\nu(\xi)\right]^{1/2} dz \\
&= \int_G \left(\int_G |(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})|^2 d\mu(x)\right)^{1/2} \times \\
&\qquad\qquad \left(\int_{\widehat{G}} |\tau(z,\xi^{-1})|^2 d\nu(\xi)\right)^{1/2} d\mu(z).
\end{aligned}
$$

So, by the Schwarz inequality and Fubini's theorem,

$$
\begin{aligned}
\|\lambda\|_{L^2(G\times\widehat{G})} &\le \left(\int_G\int_G |(\mathcal{F}_G^{-1}\sigma)(x, x\cdot z^{-1})|^2 d\mu(z)\, d\mu(x)\right)^{1/2} \times \\
&\qquad \left(\int_G\int_{\widehat{G}} |\tau(z,\xi^{-1})|^2 \nu(\xi)\, d\mu(z)\right)^{1/2}.
\end{aligned}
$$

By changing variables of integration and using Plancherel's theorem, we obtain

$$
\begin{aligned}
&\|\lambda\|_{L^2(G\times\widehat{G})} \\
&\le \left(\int_G\int_G |(\mathcal{F}_G^{-1}\sigma)(x, w)|^2 d\mu(w)\, d\mu(x)\right)^{1/2} \|\tau\|_{L^2(G\times\widehat{G})} \\
&= \left(\int_G\int_{\widehat{G}} |\sigma(x,\xi)|^2 d\nu(\xi)\, d\mu(x)\right)^{1/2} \|\tau\|_{L^2(G\times\widehat{G})} \\
&= \|\sigma\|_{L^2(G\times\widehat{G})} \|\tau\|_{L^2(G\times\widehat{G})}.
\end{aligned}
$$

□

The following theorem gives a formula for the symbol of the adjoint of a pseudo-differential operator T_σ, where $\sigma \in L^2(G\times\widehat{G})$.

Theorem 4.2 *Let σ be a symbol in $L^2(G\times\widehat{G})$. Then the adjoint of the pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is the pseudo-differential operator $T_\sigma^* : L^2(G) \to L^2(G)$ for which $T_\sigma^* = T_\tau$ and τ is the function in $L^2(G\times\widehat{G})$ given by*

$$
\tau(x,\xi) = \int_G \overline{\xi(x\cdot y^{-1})}(\mathcal{F}_G^{-1}\overline{\sigma})(y, x\cdot y^{-1})\, d\mu(y), \quad (x,\xi)\in G\times\widehat{G}.
$$

.

Proof Let $\sigma \in L^2(G \times \widehat{G})$. Then by (3.1), $T_\sigma : L^2(G) \to L^2(G)$ is a Hilbert–Schmidt operator with kernel K given by

$$K(x,y) = (\mathcal{F}_G^{-1}\sigma)(x, x \cdot y^{-1}), \quad x, y \in G.$$

So, the kernel H of $T_\sigma^* : L^2(G) \to L^2(G)$ is the function on $G \times G$ such that

$$H(x,y) = \overline{(\mathcal{F}_G^{-1}\sigma)(y, y \cdot x^{-1})}, \quad x, y \in G.$$

By (3.2), the symbol τ of $T_\sigma^* : L^2(G) \to L^2(G)$ can be written explicitly as

$$\tau(x,\xi) = \overline{\xi(x)}(\mathcal{F}_{G,2}H)(x, \xi^{-1}), \quad (x,\xi) \in G \times \widehat{G}.$$

But

$$\begin{aligned}(\mathcal{F}_{G,2}H)(x,\xi^{-1}) &= \int_G \overline{\xi^{-1}(y)} H(x,y)\, d\mu(y) \\ &= \int_G \xi(y)\xi(y)\overline{(\mathcal{F}_G^{-1}\sigma)(y, y \cdot x^{-1})} d\mu(y), \quad (x,\xi) \in G \times \widehat{G}.\end{aligned}$$

Now,

$$\begin{aligned}\overline{(\mathcal{F}_G^{-1}\sigma)(y, y \cdot x^{-1})} &= \int_{\widehat{G}} \overline{\xi(y \cdot x^{-1})}\overline{\sigma}(y,\xi)\, d\nu(\xi) \\ &= \int_{\widehat{G}} \xi(x \cdot y^{-1})\overline{\sigma}(y,\xi)\, d\nu(\xi) \\ &= (\mathcal{F}_G^{-1}\overline{\sigma})(y, x \cdot y^{-1}), \quad x, y \in G.\end{aligned}$$

Therefore

$$\begin{aligned}\tau(x,\xi) &= \overline{\xi(x)} \int_G \xi(y)(\mathcal{F}_G^{-1}\overline{\sigma})(y, x \cdot y^{-1})\, d\mu(y) \\ &= \int_G \overline{\xi(x \cdot y^{-1})}(\mathcal{F}_G^{-1}\overline{\sigma})(y, x \cdot y^{-1})\, d\mu(y), \quad (x,\xi) \in G \times \widehat{G}.\end{aligned}$$

It remains to prove that $\tau \in L^2(G \times \widehat{G})$. By changing the variable of integration from y to z via $z = x \cdot y^{-1}$, we get

$$\tau(x,\xi) = \int_G \overline{\xi(z)}(\mathcal{F}_G^{-1}\overline{\sigma})(x \cdot z^{-1}, z)\, d\mu(z), \quad (x,\xi) \in G \times \widehat{G}.$$

So, by Plancherel's theorem,

$$\begin{aligned}
\|\tau\|^2_{L^2(G\times\widehat{G})} &= \int_G \int_{\widehat{G}} |\tau(x,\xi)|^2 d\nu(\xi)\, d\mu(x) \\
&= \int_G \int_{\widehat{G}} \left| \int_G \overline{\xi(z)} (\mathcal{F}_G^{-1}\overline{\sigma})(x\cdot z^{-1}, z)\, d\mu(z) \right|^2 d\nu(\xi)\, d\mu(x) \\
&= \int_G \int_G |(\mathcal{F}_G^{-1}\overline{\sigma})(x\cdot z^{-1}, z)|^2 d\mu(z)\, d\mu(x).
\end{aligned}$$

By Fubini's theorem and a change of variables,

$$\begin{aligned}
\|\tau\|^2_{L^2(G\times\widehat{G})} &= \int_G \int_G |(\mathcal{F}_G^{-1}\overline{\sigma})(x\cdot z^{-1}, z|^2 d\mu(x)\, d\mu(z) \\
&= \int_G \int_G |(\mathcal{F}_G^{-1}\overline{\sigma})(x, z)|^2 d\mu(x)\, d\mu(z) \\
&= \int_G \int_G |(\mathcal{F}_G^{-1}\overline{\sigma})(x, z)|^2 d\mu(z)\, d\mu(x).
\end{aligned}$$

By Plancherel's theorem again,

$$\begin{aligned}
\|\tau\|^2_{L^2(G\times\widehat{G})} &= \int_G \int_{\widehat{G}} |\overline{\sigma}(x,\xi)|^2 d\nu(\xi)\, d\mu(x) \\
&= \int_G \int_{\widehat{G}} |\sigma(x,\xi)|^2 d\nu(\xi)\, d\mu(x) \\
&= \|\sigma\|^2_{L^2(G\times\widehat{G})} < \infty.
\end{aligned}$$

□

5 A New Convolution and Trace Class Operators

Theorem 4.1 gives us a new convolution of two functions in $L^2(G\times\widehat{G})$. To see what it is, let σ and τ be two functions in $L^2(G\times\widehat{G})$. Then we define the convolution $\sigma \circledast \tau$ of σ and τ to be the function in $L^2(G\times\widehat{G})$ by

$$(\sigma \circledast \tau)(x,\xi) = \int_G \xi(y)(\mathcal{F}_G^{-1}\sigma)(x, x\cdot y^{-1})\tau(y,\xi^{-1})\, d\mu(y), \quad (x,\xi)\in G\times\widehat{G}.$$

As an application of this new convolution, we give a characterization of trace class pseudo-differential operators on a locally compact, Hausdorff and abelian group G. To do this, we define W to be the set of all functions σ in $L^2(G \times \widehat{G})$ such that there exist functions α and β in $L^2(G \times \widehat{G})$ for which

$$\sigma = \alpha \circledast \beta.$$

Theorem 5.1 *Let $\sigma \in L^2(G \times \widehat{G})$. Then $T_\sigma : L^2(G) \to L^2(G)$ is a trace class pseudo-differential operator if and only if $\sigma \in W$.*

Proof Let $\sigma \in W$. Then there exist functions α and β in $L^2(G \times \widehat{G})$ such that $\sigma = \alpha \circledast \beta$. Thus, by Theorem 4.1,

$$T_\alpha T_\beta = T_{\alpha \circledast \beta} = T_\sigma.$$

Since T_α and T_β are Hilbert–Schmidt operators from $L^2(G)$ into $L^2(G)$, it follows that $T_\sigma : L^2(G) \to L^2(G)$ is a trace class operator. Conversely, suppose that $T_\sigma : L^2(G) \to L^2(G)$ is a trace class operator. Then we can find Hilbert–Schmidt operators from $L^2(G)$ into $L^2(G)$ such that $T_\sigma = AB$. Transforming the kernels of A and B into, respectively, the symbols of A and B by the formula (3.2), we get functions α and β in $L^2(G \times \widehat{G})$ for which $A = T_\alpha$ and $B = T_\beta$. Hence

$$T_\sigma = T_\alpha T_\beta = T_{\alpha \circledast \beta}.$$

Therefore $\sigma = \alpha \circledast \beta$, which is the same as saying that $\sigma \in W$. □

We can give a formula for the trace of a trace class pseudo-differential operator.

Theorem 5.2 *Let $\sigma \in L^2(G \times \widehat{G})$ be such that $T_\sigma : L^2(G) \to L^2(G)$ is a trace class pseudo-differential operator. Then*

$$\mathrm{tr}(T_\sigma) = \int_G \int_{\widehat{G}} \sigma(x, \xi)\, d\nu(\xi)\, d\mu(x).$$

Proof Since $\sigma \in L^2(G \times \widehat{G})$, it follows from (3.1) that the kernel K of $T_\sigma : L^2(G) \to L^2(G)$ is the function on $G \times G$ given by

$$K(x, y) = (\mathcal{F}_G^{-1}\sigma)(x, x \cdot y^{-1}), \quad x, y \in G.$$

Since $T_\sigma : L^2(G) \to L^2(G)$ is a trace class operator, it follows that $\int_G K(x,x)\, d\mu(x)$ exists and

$$\begin{aligned}
\mathrm{tr}(T_\sigma) &= \int_G K(x,x)\, d\mu(x) \\
&= \int_G (\mathcal{F}_G^{-1}\sigma)(x,e)\, d\mu(x) \\
&= \int_G \int_{\widehat{G}} \xi(e)\sigma(x,\xi)\, d\nu(\xi)\, d\mu(x),
\end{aligned}$$

where e denotes the identity element of the group G. Since $\xi(e) = 1$, we obtain

$$\mathrm{tr}(T_\sigma) = \int_G \int_{\widehat{G}} \sigma(x,\xi)\, d\nu(\xi)\, d\mu(x).$$

□

It is an immediate corollary of Theorem 5.1 that W is a subspace of $L^1(G \times \widehat{G})$. Furthermore, we have the following result on the space W.

Theorem 5.3 *W is a dense subspace of $L^2(G \times \widehat{G})$.*

Proof We only need to prove that W is dense in $L^2(G \times \widehat{G})$. Let F be the set of all functions σ on $G \times \widehat{G}$ such that T_σ is a finite rank operator on $L^2(G)$. Since every element in $S_2(G)$ is the limit in $S_2(G)$ of a sequence of finite rank operators on $L^2(G)$, it follows from Theorem 3.1 that F is a dense subspace of $L^2(G \times \widehat{G})$. Obviously, F is a subspace of W. Therefore W is dense in $L^2(G \times \widehat{G})$. □

Similar results for Weyl transforms, which are variants of pseudo-differential operators on $\mathbb{R}^n$, can be found in [16].

6 C^*-Algebras and H^*-Algebras

Let $\mathcal{A}$ be a Banach algebra with involution $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ and norm $\|\ \|$ such that

$$\|a^*a\| = \|a\|^2, \quad a \in \mathcal{A}.$$

Then $\mathcal{A}$ is known as a C^*-*algebra*. Suppose that the norm in $\mathcal{A}$ comes from an inner product $(\ ,\ )$ such that

$$(ab, c) = (b, a^*c), \quad a, b, c \in \mathcal{A}.$$

Then $\mathcal{A}$ is called a H^*-*algebra*. References on C^*-algebras abound. See, for instance, [2, 3]. As for H^*-algebras, the literature includes [1, 5, 6, 11, 14].

Let $S_2(G) = \{T_\sigma : L^2(G) \to L^2(G) : \sigma \in L^2(G \times \widehat{G})\}$. Then we have the following theorem.

Theorem 6.1 *$S_2(G)$ is a C^*-algebra and is also a H^*-algebra with respect to the product*

$$S_2(G) \times S_2(G) \ni (T_\sigma, T_\tau) \mapsto T_\sigma T_\tau \in S_2(G),$$

the involution

$$S_2(G) \ni T_\sigma \mapsto T_\sigma^* \in S_2(G)$$

and the norm $\| \, \|_{S_2(G)}$ given by

$$\|T_\sigma\|_{S_2(G)}^2 = \mathrm{tr}(T_\sigma T_\sigma^*), \quad T_\sigma \in S_2(G).$$

Proof To see that $S_2(G)$ is a Banach algebra with involution, we need only prove that $S_2(G)$ is complete with respect to $\| \, \|_{S_2(G)}$. Indeed, let $\{T_{\sigma_k}\}_{k=1}^\infty$ be a Cauchy sequence in $S_2(G)$. Then by Theorem 3.1, we get

$$\|\sigma_j - \sigma_k\|_{L^2(G\times\widehat{G})} = \|T_{\sigma_j} - T_{\sigma_k}\|_{S_2(G)} \to 0$$

as $j, k \to \infty$. Therefore $\{\sigma_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(G \times \widehat{G})$. Since $L^2(G\times\widehat{G})$ is complete, it follows that there exists a function $\sigma \in L^2(G\times\widehat{G})$ such that $\sigma_k \to \sigma$ in $L^2(G \times \widehat{G})$ as $k \to \infty$. By Theorem 3.1 again,

$$\|T_{\sigma_k} - T_\sigma\|_{S_2(G)} = \|\sigma_k - \sigma\|_{L^2(G\times\widehat{G})} \to 0$$

as $k \to \infty$. Therefore $T_{\sigma_k} \to T_\sigma$ in $S_2(G)$ as $k \to \infty$. Hence $S_2(G)$ is complete with respect to the norm $\| \, \|_{S_2(G)}$. Now, let $T_\sigma \in S_2(G)$. Then

$$\|T_\sigma^* T_\sigma\|_{S_2(G)}^2 = \mathrm{tr}((T_\sigma^* T_\sigma)(T_\sigma^* T_\sigma)^*) = \mathrm{tr}(T_\sigma^* T_\sigma T_\sigma^* T_\sigma) = [\mathrm{tr}(T_\sigma T_\sigma^*)]^2 = \|T_\sigma\|_{S_2(G)}^4.$$

Therefore

$$\|T_\sigma T_\sigma^*\|_{S_2(G)} = \|T_\sigma\|_{S_2(G)}^2$$

and we conclude that $S_2(G)$ is a C^*-algebra. The inner product $(\, , \,)_{S_2(G)}$ in $S_2(G)$ is given by

$$(T_\sigma, T_\tau)_{S_2(G)} = \mathrm{tr}(T_\sigma T_\tau^*), \quad T_\sigma, T_\tau \in S_2(G).$$

Furthermore, let T_α, T_β and T_γ be elements in $S_2(G)$. Then

$$\begin{aligned}(T_\alpha T_\beta, T_\gamma)_{S_2(G)} &= \mathrm{tr}((T_\alpha T_\beta)T_\gamma^*)\\ &= \mathrm{tr}(T_\beta(T_\gamma^*(T_\alpha^*)^*))\\ &= \mathrm{tr}(T_\beta(T_\alpha^* T_\gamma)^*)\\ &= (T_\beta, T_\alpha^* T_\gamma)_{S_2(G)}.\end{aligned}$$

Therefore $S_2(G)$ is a H^*-algebra. □

We can equip $L^2(G \times \widehat{G})$ with the product

$$L^2(G \times \widehat{G}) \times L^2(G \times \widehat{G}) \ni (\sigma, \tau) \mapsto \sigma \circledast \tau \in L^2(G \times \widehat{G}) \tag{6.1}$$

and the involution

$$L^2(G \times \widehat{G}) \ni \sigma \mapsto \sigma^* \in L^2(G \times \widehat{G}), \tag{6.2}$$

where

$$\sigma^*(x, \xi) = \int_G \overline{\xi(x \cdot y^{-1})}(\mathcal{F}_G^{-1}\overline{\sigma})(y, x \cdot y^{-1})\, d\mu(y), \quad (x, \xi) \in G \times \widehat{G}.$$

Then by Theorem 4.1 on the product, Theorem 4.2 on the adjoint, and Theorem 6.1 on the C^*-algebra and H^*-algebra of pseudo-differential operators, we have the following theorem.

Theorem 6.2 *$L^2(G \times \widehat{G})$ is a C^*-algebra, which is also a H^*-algebra, with respect to the multiplication $\circledast$ and the involution $*$ defined by* (6.1) *and* (6.2) *respectively.*

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