Characterizations of Self-Adjointness, Normality, Invertibility and Unitarity of Pseudo-Differential Operators on Compact and Hausdorff Groups

Majid Jamalpourbirgani and M. W. Wong

Abstract. We give explicit formulas for the adjoint, product and inverse of a bounded pseudo-differential operator in terms of its symbol on a compact and Hausdorff group. As applications we give necessary and sufficient conditions to insure that a bounded pseudo-differential operator on a compact and Hausdorff group G is self-adjoint, normal and unitary on $L^2(G)$, and invertible on $L^p(G)$ for $1 \leq p < \infty$.

Mathematics Subject Classification (2000). Primary 47F05, 47G30; Secondary 35J70.

Keywords. compact and Hausdorff groups, symbols, pseudo-differential operators, self-adjoint, normal, invertible, unitary.

1. Introduction

Let G be a compact and Hausdorff group on which the left (and right) Haar measure is denoted by μ. Let ξ be an irreducible and unitary representation of G on a complex and separable Hilbert space X_ξ. Since G is compact, it is well known that X_ξ is finite-dimensional. We let d_ξ be the dimension of X_ξ. The number d_ξ is also known as the degree of the representation ξ of G on X_ξ. Let $\widehat{G}$ be the set of all (equivalence classes) of irreducible and unitary representations of G, which is usually referred to as the *dual group* of G.

This work of the first-named author was supported by the International Visiting Research Traineeship (IVRT) sponsored by Professor M. W. Wong at York University in 2017. The research of the second-named author has been supported by the Natural Sciences and Engineering Research Council of Canada under Discovery Grant 0008562.

Let $f \in L^p(G)$, $p \geq 1$. Then we define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(\xi) = \int_G f(x)\xi(x)^* dx, \quad \xi \in \widehat{G}.$$

It is also well known that the Fourier inversion formula states that for a good class of functions in $L^p(G)$, $p \geq 1$,

$$f(x) = \sum_{\xi \in \widehat{G}} d_\xi \mathrm{tr}(\xi(x)\hat{f}(\xi)), \quad x \in G.$$

The Fourier inversion formula can be looked at as a formula for the identity operator on $L^p(G)$, $p \geq 1$, and as such, is a perfect symmetry that gives us the identity operator on a suitable class of functions on G.

Good references for abstract harmonic analysis abound. See, for instance, [1] [4] and [6] for abstract harmonic analysis in general and group representations, the dual group and the Fourier inversion formula in particular.

In order to obtain more interesting operators than the identity operator, we need to break the symmetry using *symbols* σ defined on the *phase space* $G \times \widehat{G}$. To wit, let σ be a *suitable* function defined on $G \times \widehat{G}$. Then for every point $(x, \xi) \in G \times \widehat{G}$, $\sigma(x, \xi)$ is a $d_\xi \times d_\xi$ matrix. For all $\xi \in \widehat{G}$, we denote by $M_{d_\xi}(\mathbb{C})$ the set of all $d_\xi \times d_\xi$ matrices with complex entries. A symbol σ on $G \times \widehat{G}$ in this paper is understood to be a mapping

$$G \times \widehat{G} \ni (x, \xi) \mapsto \sigma(x, \xi) \in M_{d_\xi}(\mathbb{C}).$$

We define the *pseudo-differential operator* T_σ on G with symbol σ by

$$(T_\sigma f)(x) = \sum_{\xi \in \widehat{G}} d_\xi \mathrm{tr}(\xi(x)\sigma(x, \xi)\hat{f}(\xi)), \quad x \in G.$$

The focus of this paper is on the functional analysis of bounded pseudo-differential operators on compact and Hausdorff groups. More explicitly, the overarching hypothesis is the boundedness of a pseudo-differential operator from $L^{p_1}(G)$ into $L^{p_2}(G)$, where $1 \leq p_1, p_2 < \infty$. Very general conditions on the boundedness of pseudo-differential operators can be found in [2] for compact Lie groups and in [8] for the unit circle centered at the origin. It should also be noted that the analysis of pseudo-differential operators on compact and Hausdorff groups can be found in [3, 5]. Results on operators related to pseudo-differential operators in the context of locally compact and Hausdorff groups can be found in [7]. Developing pseudo-differential operators at the level of topological groups rather than Lie groups manifests the many-faceted connections of these operators with mainstream areas of mathematics besides partial differential equations.

In Section 2 of the paper, we first give the result that every bounded linear operator $A : L^p(G) \to L^p(G)$, $1 \leq p < \infty$, is a pseudo-differential operator of which the symbol can be uniquely determined. This immediately implies that the mapping of symbols to pseudo-differential operators on compact and Hausdorff groups is injective. We give in Section 3 a formula for the symbols of the adjoints of bounded pseudo-differential operators from $L^{p_1}(G)$ into $L^{p_2}(G)$, $1 \le p_1, p_2 < \infty$. In particular, we give a criterion for the self-adjointness, or equivalently, the non-self-adjointness of bounded pseudo-differential operators on $L^2(G)$. We give in Section 4 a formula for the product of two pseudo-differential operators on G. As an application, a criterion for a pseudo-differential operator on G to be normal is given. In Section 5, we give results on the invertibility of pseudo-differential operators on G. In particular, we give a necessary and sufficient condition for a pseudo-differential operator on G to be invertible. A criterion for the unitarity of pseudo-differential operators on G is also given.

2. Injectivity

We begin with the result that every bounded linear operator on $L^p(G)$, $1 \le p < \infty$, is a pseudo-differential operator from $L^p(G)$ into $L^p(G)$.

Theorem 2.1. *Let : $A : L^p(G) \to L^p(G)$ be a bounded linear operator, where $1 \le p < \infty$. Then $A : L^p(G) \to L^p(G)$ is a pseudo-differential operator $T_\sigma : L^p(G) \to L^p(G)$ such that*

$$(Af)(x) = (T_\sigma f)(x) = \sum_{\xi \in \widehat{G}} d_\xi \operatorname{tr}(\xi(x)\sigma(x,\xi)\hat{f}(\xi)), \quad x \in G,$$

where

$$\sigma(x,\xi) = \xi(x)^* a(x,\xi)$$

with

$$a(x,\xi)_{nm} = (A\xi_{nm})(x)$$

for all $(x,\xi) \in G \times \widehat{G}$ and all $1 \le n, m \le d_\xi$.

Proof Let $f \in C(G)$. Then for all $\eta \in \widehat{G}$ and all positive integers j and k with $1 \le j, k \le d_\eta$,

$$\begin{aligned} \widehat{(Af)}(\eta)_{kj} &= \int_G \overline{\eta(x)_{jk}}(Af)(x)\,d\mu(x) \\ &= \int_G \overline{(A^*\eta_{jk})(x)} f(x)\,d\mu(x) \\ &= \sum_{\xi \in \widehat{G}} \sum_{m,n=1}^{d_\xi} d_\xi \hat{f}(\xi)_{mn} \int_G \overline{(A^*\eta_{jk})(x)} \xi(x)_{nm} d\mu(x) \\ &= \left[\sum_{\xi \in \widehat{G}} \sum_{m,n=1}^{d_\xi} d_\xi \hat{f}(\xi)_{mn} (A\xi_{nm})(\cdot) \right]^\wedge (\eta)_{kj}. \end{aligned}$$

So, for all $x \in G$,

$$\begin{aligned}(Af)(x) &= \sum_{\xi \in \widehat{G}} \sum_{m,n=1}^{d_\xi} d_\xi \hat{f}(\xi)_{mn} (A\xi_{nm})(x) \\ &= \sum_{\xi \in \widehat{G}} d_\xi \mathrm{tr}(\xi(x)\xi(x)^* a(x,\xi)\hat{f}(\xi)) \\ &= (T_\sigma f)(x).\end{aligned}$$

Therefore $A = T_\sigma$ and the proof is complete. □

We can now show that the mapping of symbols on $G \times \widehat{G}$ into pseudo-differential operators on G is injective.

Corollary 2.2. *Let σ and τ be symbols on $G \times \widehat{G}$ such that $T_\sigma : L^p(G) \to L^p(G)$ and $T_\tau : L^p(G) \to L^p(G)$ are bounded linear operators, where $1 \le p < \infty$. Then $\sigma = \tau$.*

3. Adjoints

We need the following lemma.

Lemma 3.1. *Let σ be a function on $G \times \widehat{G}$ such that the corresponding pseudo-differential operator $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is a bounded linear operator, where $1 \le p_1, p_2 < \infty$. Then for all positive integers m and n with $1 \le m, n \le d_\xi$,*

$$(\xi(\cdot)\sigma(\cdot,\xi))_{mn} \in L^{p_2}(G).$$

Proof There exists a positive constant C such that

$$\|T_\sigma f\|_{L^{p_2}(G)} \le C\|f\|_{L^{p_1}(G)}, \quad f \in L^{p_1}(G).$$

Now, let $\xi \in \widehat{G}$ and let m and n be positive integers such that $1 \le m, n \le d_\xi$. Then we define the function f on G by

$$f(x) = \xi(x)_{mn}, \quad x \in G.$$

We note that

$$\begin{aligned}T_\sigma \xi(\cdot)_{mn} &= \int_G d_\eta \sum_{\eta \in \widehat{G}} \sum_{j,k=1}^{d_\eta} (\eta(\cdot)\sigma(\cdot,\eta))_{jk} \overline{\eta(y)}_{jk} \xi(y)_{mn} d\mu(y) \\ &= d_\xi (\xi(\cdot)\sigma(\cdot,\xi))_{mn}.\end{aligned}$$

So,

$$(\xi(\cdot)\sigma(\cdot,\xi))_{mn} = \frac{1}{d_\xi} T_\sigma \xi(\cdot)_{mn}. \tag{3.1}$$

Since $f = \xi(\cdot)_{mn} \in L^{p_1}(G)$, it follows that

$$\|\xi(\cdot)\sigma(\cdot,\xi)\|_{L^{p_2}(G)} = \frac{1}{d_\xi}\|T_\sigma \xi(\cdot)_{mn}\|_{L^{p_2}(G)} \le \frac{1}{d_\xi}\|\xi(\cdot)_{mn}\|_{L^{p_1}(G)} < \infty$$

and this completes the proof. □

The followin theorem gives a formula for the adjoint of a bounded pseudo-differential operator on G.

Theorem 3.2. *Let σ be a symbol such that the pseudo-differential operator $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ is a bounded linear operator for $1 \le p_1, p_2 < \infty$. Then its adjoint is the pseudo-differential operator $T_\tau : L^{p_2'}(G) \to L^{p_1'}(G)$, where*

$$\tau(x,\xi) = \xi(x)^* \sum_{\gamma\in\widehat{G}} \frac{d_\gamma^2}{d_\xi}(\mathrm{tr}[\gamma(x)(\gamma(y)\sigma(y,\gamma))^*])^\wedge(\xi^*), \quad (x,\xi)\in G\times\widehat{G}.$$

Proof Let γ and ξ be elements in $\widehat{G}$. Then for all positive integers t, m, n and l with $1 \le t, m \le d_\gamma$ and $1 \le n, l \le d_\xi$,

$$\begin{aligned}\int_G (\gamma(y)\sigma(y,\gamma))_{tm}\overline{\xi(y)}_{nl}d\mu(y) &= \int_G \frac{1}{d_\gamma}(T_\sigma\gamma_{tm})(y)\overline{\xi(y)}_{nl}d\mu(y)\\ &= \int_G \frac{1}{d_\gamma}\gamma(y)_{tm}\overline{(T_\tau\xi_{nl})(y)}d\mu(y)\\ &= \int_G \frac{d_\xi}{d_\gamma}\gamma(y)_{tm}\overline{(\xi(y)\tau(y,\xi))}_{nl}d\mu(y), \quad (3.2)\end{aligned}$$

and hence

$$\overline{\int_G (\gamma(y)\sigma(y,\gamma))_{tm}\overline{\xi(y)}_{nl}d\mu(y)} = \frac{d_\xi}{d_\gamma}\int_G (\xi(y)\tau(y,\xi))_{nl}\overline{\gamma(y)}_{tm}d\mu(y).$$

Now, using the Fourier transform on G and (3.1), we get

$$\overline{[((\gamma(\cdot)\sigma(\cdot,\gamma))_{tm})^\wedge(\xi)]_{ln}} = \frac{d_\xi}{d_\gamma}[((\xi(\cdot)\tau(\cdot,\xi))_{nl})^\wedge(\gamma)]_{mt}. \quad (3.3)$$

It follows from the Fourier inversion formula on G and (3.3) that for all $(x,\xi) \in G\times\widehat{G}$ and $1 \le n, l \le d_\xi$,

$$\begin{aligned}((\xi(x)\tau(x,\xi))_{nl} &= \sum_{\gamma\in\widehat{G}} d_\gamma \mathrm{tr}(\gamma(x)((\xi(\cdot)\tau(\cdot,\xi))_{nl})^\wedge(\gamma))\\ &= \sum_{\gamma\in\widehat{G}}\sum_{t,m=1}^{d_\gamma} d_\gamma\gamma(x)_{tm}[((\xi(\cdot)\tau(\cdot,\xi))_{nl})^\wedge(\gamma)]_{mt}\\ &= \sum_{\gamma\in\widehat{G}}\sum_{t,m=1}^{d_\gamma}\frac{d_\gamma^2}{d_\xi}\gamma(x)_{tm}\overline{[((\gamma(\cdot)\sigma(\cdot,\gamma))_{tm})^\wedge(\xi)]_{ln}}.\end{aligned}$$

Using the Fourier transform on G, we get

$$\begin{aligned}
((\xi(x)\tau(x,\xi))_{nl} &= \sum_{\gamma\in\widehat{G}}\sum_{t,m=1}^{d_\gamma}\frac{d_\gamma^2}{d_\xi}\gamma(x)_{tm}\int_G \overline{((\gamma(y)\sigma(y,\gamma))_{tm}}\xi(y)_{nl}d\mu(y)\\
&= \sum_{\gamma\in\widehat{G}}\sum_{t,m=1}^{d_\gamma}\frac{d_\gamma^2}{d_\xi}\int_G \gamma(x)_{tm}[(\gamma(y)\sigma(y,\gamma))^*]_{mt}\xi(y)_{nl}d\mu(y)\\
&= \sum_{\gamma\in\widehat{G}}\frac{d_\gamma^2}{d_\xi}\int_G \mathrm{tr}[\gamma(x)(\gamma(y)\sigma(y,\gamma))^*]\xi(y)_{nl}d\mu(y)\\
&= \sum_{\gamma\in\widehat{G}}\frac{d_\gamma^2}{d_\xi}((\mathrm{tr}[\gamma(x)(\gamma(y)\sigma(y,\gamma))^*])^\wedge(\xi^*)_{nl}.
\end{aligned}$$

Therefore

$$\tau(x,\xi) = \xi(x)^*\sum_{\gamma\in\widehat{G}}\frac{d_\gamma^2}{d\xi}((\mathrm{tr}[\gamma(x)(\gamma(y)\sigma(y,\gamma))^*])^\wedge(\xi^*),\quad (x,\xi)\in G\times\widehat{G}.$$

□

A criterion for the self-adjointness of bounded pseudo-differential operators on G is provided by the following theorem.

Theorem 3.3. *Let σ be a symbol on $G\times\widehat{G}$. Then the pseudo-differential operator $T_\sigma : L^2(G)\to L^2(G)$ is self-adjoint if and only if for all γ and ξ in $\widehat{G}$ and all positive integers t, m, n and l with $1\le t,m\le d\gamma$ and $1\le n,l\le d_\xi$,*

$$d_\gamma\int_G(\gamma(y)\sigma(y,\gamma))_{tm}\overline{\xi(y)_{nl}}d\mu(y) = d_\xi\int_G\gamma(y)_{tm}\overline{\xi(y)\sigma(y,\xi))_{nl}}d\mu(y).$$

Proof Suppose that $T_\sigma : L^2(G)\to L^2(G)$ is self-adjoint. Then for all $(y,\xi)\in G\times\widehat{G}$ and all positive integers n and l with $1\le n,l\le d_\xi$,

$$(\xi(y)\sigma(y,\xi))_{nl} = (\xi(y)\tau(y,\xi))_{nl},$$

where τ is the symbol of the adjoint of $T_\sigma : L^2(G)\to L^2(G)$. By (3.2), we get for all γ and ξ in $\widehat{G}$ and all positive integers t, m, n and l with $1\le t,m\le d_\gamma$ and $1\le n,l\le d_\xi$,

$$d_\gamma\int_G(\gamma(y)\sigma(y,\gamma))_{tm}\overline{\xi(y)_{nl}}d\mu(y) = d_\xi\int_G\gamma(y)_{tm}\overline{\xi(y)\sigma(y,\xi))_{nl}}d\mu(y).$$

Conversely, suppose that

$$d_\gamma\int_G(\gamma(y)\sigma(y,\gamma))_{tm}\overline{\xi(y)_{nl}}d\mu(y) = \int_G\gamma(y)_{tm}\overline{(\xi(y)\sigma(y,\xi))_{nl}}d\mu(y)$$

for all γ and ξ in $\widehat{G}$ and all positive integers t, m, n and l with $1 \le t, m \le d_\gamma$ and $1 \le n, l \le d_\xi$. Then as in the proof of Theorem 3.2, we get

$$\tau(x,\xi) = \sigma(x,\xi)$$

for all $(x,\xi) \in G \times \widehat{G}$, where τ is the symbol of the adjoint of $T_\sigma : L^2(G) \to L^2(G)$. Therefore $T_\sigma : L^2(G) \to L^2(G)$ is self-adjoint. □

4. Products

The basic formula for the symbol of the product of two bounded pseudo-differential operators on G is the content of the following theorem.

Theorem 4.1. *Let σ and τ be symbols on $G \times \widehat{G}$ such that $T_\sigma : L^{p_1}(G) \to L^{p_2}(G)$ and $T_\tau : L^{p_2}(G) \to L^{p_3}(G)$ be bounded linear operators, where $1 \le p_1, p_2, p_3 < \infty$. Then $T_\sigma T_\tau : L^{p_1}(G) \to L^{p_3}(G)$ is the pseudo-differential operator $T_\lambda : L^{p_1}(G) \to L^{p_3}(G)$, where λ is the symbol on $G \times \widehat{G}$ given by*

$$\lambda(x,\xi) = \xi(x)^* \sum_{\omega \in \widehat{G}} d_\omega \int_G \mathrm{tr}[\omega(x)((\sigma^*(y,\omega))^*\omega(y)^*]\xi(y)\tau(y,\xi)\, d\mu(y)$$

for all $(x,\xi) \in G \times \widehat{G}$.

Proof By Theorem 2.1, wee see that for all elements ξ and ω in $\widehat{G}$ and all positive integers n, m, k and l with $1 \le n, m \le d_\xi$ and $1 \le k, l \le d_\omega$, we have

$$\begin{aligned}
& \int_G (\xi(y)\lambda(y,\xi))_{mn}\overline{\omega(y)_{kl}} d\mu(y) \\
= & \int_G \frac{1}{d_\xi}(T_\sigma T_\tau \xi_{mn})(y)\overline{\omega(y)_{kl}} d\mu(y) \\
= & \int_G \frac{1}{d_\xi}(T_\tau \xi_{mn})(y)\overline{(T_\sigma^* \omega_{kl})(y)} d\mu(y) \\
= & \int_G d_\omega(\xi(y)\tau(y,\xi))_{mn}(((\sigma^*(y,\omega))^*\omega(y)^*)_{lk} d\mu(y).
\end{aligned}$$

So,

$$((\xi(\cdot)\lambda(\cdot,\xi))^\wedge_{mn}(\omega)_{lk} = \int_G d_\omega(\xi(y)\tau(y,\xi))_{mn}((\sigma^*(y,\omega))^*\omega(y)^*)_{lk} d\mu(y).$$

Therefore for all $(x,\xi) \in G \times \widehat{G}$, we get by the Fourier inversion formula on G

$$\begin{aligned}
& ((\xi(x)\lambda(x,\xi))_{mn} \\
= & \sum_{\omega \in \widehat{G}} d_\omega \mathrm{tr}[\omega(x)((\xi(\cdot)\lambda(\cdot,\xi))_{mn})^\wedge(\omega)] \\
= & \sum_{\omega \in \widehat{G}} \sum_{k,l=1}^{d_\omega} \omega(x)kl[((\xi(\cdot)\lambda(\cdot,\xi))_{mn})^\wedge(\omega)]_{lk} \\
= & \sum_{\omega \in \widehat{G}} \sum_{k,l=1}^{d_\xi} d_\omega \omega(x)_{kl} \int_G (\xi(y)\tau(y,\xi))_{mn}((\sigma^*(y,\omega)^*)\omega(y)^*)_{lk} d\mu(y) \\
= & \sum_{\omega \in \widehat{G}} d_\omega \int_G \mathrm{tr}[\omega(x)(\sigma^*(y,\omega))^*\omega(y)^*](\xi(y)\tau(y,\xi))_{mn} d\mu(y).
\end{aligned}$$

Then for all $(x,\xi) \in G \times \widehat{G}$,

$$\xi(x)\lambda(x,\xi) = \sum_{\omega \in \widehat{G}} d_\omega \int_G \mathrm{tr}[\omega(x)(\sigma^*(y,\omega)^*\omega(y)^*]\xi(y)\tau(y,\xi)\, d\mu(y)$$

and hence

$$\lambda(x,\xi) = \xi(x)^* \sum_{\omega \in \widehat{G}} \int_G \mathrm{tr}[\omega(x)(\sigma^*(y,\omega)^*)\omega(y)^*](\xi(y)\tau(y,\xi))\, d\mu(y).$$

□

The following two theorems give, respectively, a characterization of the normality and unitarity of bounded pseudo-differential operators on G.

Theorem 4.2. *Let σ be a symbol on $G \times \widehat{G}$ be such that the corresponding pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is a bounded linear operator. Then $T_\sigma : L^2(G) \to L^2(G)$ is normal if and only if for all ξ and ω in $\widehat{G}$ with $1 \le m,n \le d_\xi$ and $1 \le l,k \le d_\omega$,*

$$\begin{aligned}
& \int_G (\xi(y)\sigma(y,\xi))_{mn} \overline{(\omega(y)\sigma(y,\omega))_{lk}} d\mu(y) \\
= & \int_G (\xi(y)\sigma^*(y,\xi))_{mn} \overline{(\omega(y)(\sigma^*(y,\omega))_{lk}} d\mu(y).
\end{aligned}$$

Proof Suppose that $T_\sigma : L^2(G) \to L^2(G)$ is a normal operator. Then for all ξ and ω in $\widehat{G}$ with $1 \le m, n \le d_\xi$ and $1 \le l, k \le d_\omega$,

$$\begin{aligned}
& \int_G (\xi(y)\sigma(y,\xi))_{mn}\overline{(\omega(y)\sigma(y,\omega))_{lk}}d\mu(y) \\
= \quad & \frac{1}{d_\xi d_\omega}(T_\sigma \xi_{mn})(y)\overline{(T_\sigma \omega_{lk})(y)}d\mu(y) \\
= \quad & \int_G \frac{1}{d_\xi d_\omega}(T_\sigma^* T_\sigma \xi_{mn})(y)\overline{\omega_{lk}(y)}d\mu(y) \\
= \quad & \int_G \frac{1}{d_\xi d_\omega}(T_\sigma T_\sigma^* \xi_{mn})(y)\overline{\omega_{lk}(y)}d\mu(y) \\
= \quad & \int_G \frac{1}{d_\xi d_\omega}(T_\sigma^* \xi_{mn})(y)\overline{(T_\sigma^* \omega_{lk})(y)}d\mu(y) \\
= \quad & \int_G (\xi(y)\sigma^*(y,\xi))_{mn}\overline{(\omega(y)\sigma^*(y,\omega(y))_{lk}}d\mu(y).
\end{aligned}$$

Conversely, suppose that

$$\begin{aligned}
& \int_G (\xi(y)\sigma(y,\xi))_{mn}\overline{(\omega(y)\sigma(y,\omega))_{lk}}d\mu(y) \\
= \quad & \int_G (\xi(y)\sigma^*(y,\xi))_{mn}\overline{(\omega(y)\sigma^*(y,\omega))_{lk}}d\mu(y)
\end{aligned}$$

for all ξ and ω in $\widehat{G}$ with $1 \le m, n \le d_\xi$ and $1 \le l, k \le d_\omega$. Then for all $x \in G$,

$$\begin{aligned}
& \omega_{lk}(x)\int_G (\xi(y)\sigma(y,\xi))_{mn}\overline{(\omega(y)\sigma(y,\xi))_{lk}}d\mu(y) \\
= \quad & \omega_{lk}(x)\int_G (\xi(y)\sigma^*(y,\xi))_{mn}\overline{(\omega(y)\sigma^*(y,\omega))_{lk}}d\mu(y)
\end{aligned}$$

and so,

$$\begin{aligned}
& \int_G \operatorname{tr}[\omega(x)\sigma(y,\omega)^*\omega(y)^*](\xi(y)\sigma(y,\omega))_{mn}d\mu(y) \\
= \quad & \int_G \operatorname{tr}[\omega(x)(\sigma^*(y,\omega))^*\omega(y)^*](\xi(y)\sigma^*(y,\xi))_{mn}d\mu(y).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{\omega\in\widehat{G}} d_\omega \int_G \operatorname{tr}[\omega(x)\sigma(y,\omega)^*\omega(y)^*](\xi(y)\sigma(y,\xi))_{mn}d\mu(y) \\
= \quad & \sum_{\omega\in\widehat{G}} d_\omega \int_G \operatorname{tr}[\omega(x)(\sigma^*(y,\omega))^*\omega(y)^*](\xi(y)\sigma^*(y,\xi))_{mn}d\mu(y)
\end{aligned}$$

and hence for all $x \in G$,

$$\begin{aligned} & \xi(x)^* \sum_{\omega \in \widehat{G}} d_\omega \in_G \operatorname{tr}[\omega(x)\sigma(y,\omega)^*\omega(y)^*]\xi(y)\sigma(y,\xi)\, d\mu(y) \\ = \; & \xi(x)^* \sum_{\omega \in \widehat{G}} d_\omega \int_G \operatorname{tr}[\omega(x)(\sigma^*(y,\omega))^*\omega(y)^*]\xi(y)\sigma^*(y,\xi)\, d\mu(y). \end{aligned}$$

By Theorem 4.1, the symbol of $T_\sigma T_\sigma^*$ is equal to the symbol of $T_\sigma^* T_\sigma$. Therefore

$$T_\sigma T_\sigma^* = T_\sigma^* T_\sigma$$

and the proof is complete. □

Theorem 4.3. *Let σ be a symbol on $G \times \widehat{G}$ such that the corresponding pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is a bounded linear operator. Then $T_\sigma : L^2(G) \to L^2(G)$ is unitary if and only if for all ξ and ω in $\widehat{G}$ with $1 \le m, n \le d_\xi$ and $1 \le l, k \le d_\omega$,*

$$\begin{aligned} & \int_G (\xi(y)\sigma(y,\xi))_{mn}\overline{(\omega(y)\sigma(y,\omega))_{lk}}\, d\mu(y) \\ = \; & \int_G (\xi(y)\sigma^*(y,\xi))_{mn}\overline{(\omega(y)\sigma^*(y,\omega))_{lk}}\, d\mu(y) \\ = \; & \begin{cases} 0, & \xi \ne \omega, \\ 0, & m \ne k \text{ or } n \ne l, \\ \frac{1}{d_\xi}, & \xi = \omega,\ m = k,\ n = l. \end{cases} \end{aligned}$$

Proof Suppose that $T_\sigma : L^2(G) \to L^2(G)$ is unitary. Then for all elements ξ and ω in $\widehat{G}$ with $1 \le m, n \le d_\xi$ and $1 \le l, k \le d_\omega$, we get by Theorem 4.2 and the Peter–Weyl theorem to the effect that $\{\sqrt{d_\xi}\xi_{mn} : 1 \le m, n \le d_\xi, \xi \in \widehat{G}\}$ is an orthonormal basis for $L^2(G)$,

$$\begin{aligned}
& \int_G (\xi(y)\sigma^*(y,\xi))_{mn}\overline{(\omega(y)\sigma^*(y,\xi))_{lk}}d\mu(y) \\
= & \int_G (\xi(y)\sigma(y,\xi))_{mn}\overline{(\omega(y)\sigma(y,\omega))_{lk}}d\mu(y) \\
= & \int_G \frac{1}{d_\xi d_\omega}(T_\sigma\xi_{mn})(y)\overline{(T_\sigma\omega_{lk})(y)}d\mu(y) \\
= & \int_G \frac{1}{d_\xi d_\omega}(T_\sigma^* T_\sigma\xi_{mn})(y)\overline{\omega_{lk}(y)}d\mu(y) \\
= & \int_G \xi_{mn}(y)\overline{\omega_{lk}(y)}d\mu(y) \\
= & \begin{cases} 0, & \xi \neq \omega, \\ 0, & m \neq k \text{ or } n \neq l, \\ \frac{1}{d_\xi}, & \xi = \omega,\ m = k,\ n = l. \end{cases}
\end{aligned}$$

For the converse, let $x \in G$. Then for all elements ξ and ω in $\widehat{G}$ with $1 \le m, n \le d_\xi$ and $1 \le l, k \le d_\omega$,

$$\begin{aligned}
& \omega_{lk}(x)\int_G (\xi(y)\sigma(y,\xi))_{mn}\overline{(\omega(y)\sigma(y,\omega))_{lk}}d\mu(y) \\
= & \omega_{lk}(x)\int_G\int_G (\xi(y)\sigma^*(y,\xi))_{mn}\overline{(\omega(y)\sigma^*(y,\omega))_{lk}}d\mu(y) \\
= & \omega_{lk}(x)\int_G \xi_{mn}(y)\overline{\omega_{lk}(y)}d\mu(y)
\end{aligned}$$

and so,

$$\begin{aligned}
& \int_G \mathrm{tr}[\omega(x)\sigma(y,\omega)^*\omega(y)^*](\xi(y)\sigma(y,\xi))_{mn}d\mu(y) \\
= & \int_G \mathrm{tr}[\omega(x)(\sigma^*(y,w))^*\omega(y)^*](\xi(y)\sigma^*(y,\xi))_{mn}d\mu(y) \\
= & \int_G \mathrm{tr}[\omega(x)\omega(y)^*]\xi_{mn}(y)\, d\mu(y).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{\omega\in\widehat{G}} d_\omega \int_G \mathrm{tr}[\omega(x)\sigma(y,\omega)^*\omega(y)^*](\xi(y)\sigma(y,\xi))_{mn}d\mu(y) \\
= & \sum_{\omega\in\widehat{G}} d_\omega \int_G \mathrm{tr}[\omega(x)(\sigma^*(y,\omega))^*\omega(y)^*](\xi(y)\sigma^*(y,\xi))_{mn}d\mu(y) \\
= & \sum_{\omega\in\widehat{G}} d_\omega \int_G \mathrm{tr}[\omega(x)\omega(y)^*]\xi_{mn}(y)\, d\mu(y)
\end{aligned}$$

and hence

$$\begin{aligned}
& \xi(x)^* \sum_{\omega\in\widehat{G}} d_\omega \in_G \operatorname{tr} \int_G \operatorname{tr}[\omega(x)\sigma(y,\omega)^*\omega(y)^*]\xi(y)\sigma(y,\xi))\, d\mu(y) \\
= & \ \xi(x)^* \sum_{\omega\in\widehat{G}} d_\omega \int_G \operatorname{tr}[\omega(x)(\sigma^*(y,\omega))^*\omega(y)^*](\xi(y)\sigma^*(y,\xi))\, d\mu(y) \\
= & \ \xi(x)^* \sum_{\omega\in\widehat{G}} d_\omega \int_G \operatorname{tr}[\omega(x)\omega(y)^*]\xi(y)\, d\mu(y) \\
= & \ \xi(x)^* \int_G \delta(x\cdot y^{-1})\xi(y)\, d\mu(y) \\
= & \ \xi(x)^*\xi(x) \\
= & \ I,
\end{aligned}$$

where I is the identity matrix of order d_ξ. Thus, by Theorem 4.1,

$$T_\sigma T_\sigma^* = T_\sigma^* T_\sigma = I$$

and this proves that $T_\sigma : L^2(G) \to L^2(G)$ is unitary. □

Theorem 4.4. *Let σ be a symbol on $G \times \widehat{G}$. Then the corresponding pseudo-differential operator $T_\sigma : l^2(G) \to L^2(G)$ is unitary if and only if*

$$\{\sqrt{d_\xi}(\xi(\cdot)\sigma(\cdot,\xi))_{mn} : 1 \le m,n \le d_\xi, \xi \in \widehat{G}\}$$

and

$$\{\sqrt{d_\xi}(\xi(\cdot)\sigma^*(\cdot,\xi))_{m,n} : 1 \le m,n \le d_\xi, \xi \in \widehat{G}\}$$

are orthonormal bases for $L^2(G)$.

Proof Suppose that $T_\sigma : L^2(G) \to L^2(G)$ is unitary. Then $T_\sigma : L^2(G) \to L^2(G)$ is invertible and hence surjective. So, for all $f \in L^2(G)$, there exists a function $g \in L^2(G)$ such that

$$T_\sigma g = \sum_{\xi\in\widehat{G}} \sum_{m,n=1}^{d_\xi} d_\xi (\xi(\cdot)\sigma(\cdot,\xi))_{mn}\hat{g}(\xi)_{nm} = f.$$

By Theorem 4.3, we get for all elements ξ and η in $\widehat{G}$ and all positive integers m, n, k and l with $1 \le m,n \le d_\xi$ and $1 \le k,l \le d_\eta$,

$$\int_G \xi(x)_{mn}\overline{\eta(x)_{kl}} d\mu(x) = \int_G (\xi(x)\sigma(x,\xi))_{mn}\overline{(\eta(x)\sigma(x,\eta))_{kl}} d\mu(x).$$

We know by the Peter–Weyl theorem that $\{\sqrt{d_\omega}\omega_{mn} : 1 \le m,n \le d_\omega, \omega \in \widehat{G}\}$ is an orthonormal basis for $L^2(G)$. So, the set $\{\sqrt{d_\xi}(\xi(\cdot)\sigma(\cdot,\xi))_{mn} : 1 \le m,n \le d_\xi\}$ is orthonormal. Since $T_\sigma^* : L^2(G) \to L^2(G)$ is also unitary, it follows that $\{\sqrt{d_\xi}(\xi(\cdot)\sigma^*(\cdot,\xi))_{mn} : 1 \le m,n \le d_\xi, \xi \in \widehat{G}\}$ is an orthonormal basis for $L^2(G)$. The converse follows immediately from Theorem 4.3. □

5. Invertibility

A necessary and sufficient condition for a bounded pseudo-differential operator on G to be invertible is first given.

Theorem 5.1. *Let σ be a symbol on $G \times \widehat{G}$ such that the corresponding pseudo-differential operator $T_\sigma : L^p(G) \to L^p(G)$ is a bounded linear operator for $1 \leq p < \infty$. Then $T_\sigma : L^p(G) \to L^p(G)$ is invertible if and only if there exists a symbol τ on $G \times \widehat{G}$ corresponding to a bounded pseudo-differential operator T_σ such that for all elements ξ and η in $\widehat{G}$ and all positive integers m, n, k and l with $1 \leq m, n \leq d_\xi$ and $1 \leq k, l \leq d_\eta$,*

$$\begin{aligned} & \int_G (\xi(x)\tau(x,\xi))_{mn} \overline{(\eta(x)\sigma^*(x,\eta))_{kl}} d\mu(x) \\ = & \int_G (\xi(x)\sigma(x,\xi))_{mn} \overline{(\eta(x)\tau^*(x,\eta))_{kl}} d\mu(x) \\ = & \begin{cases} 0, & \xi \neq \eta, \\ 0, & m \neq k \text{ or } m \neq l, \\ \frac{1}{d_\xi}, & \xi = \eta, \ m = k, \ n = l. \end{cases} \end{aligned}$$

In this case, $T_\sigma^{-1} = T_\tau$.

Proof Suppose that $T_\sigma : L^p(G) \to L^p(G)$ is an invertible operator. Then for all $f \in L^p(G)$ and $g \in L^{p'}(G)$,

$$(f, g) = (T_\sigma T_\sigma^{-1} f, g) = (T_\sigma^{-1} f T_\sigma^* g)$$

and so,

$$\int_G f(x)\overline{g(x)}\, d\mu(x) = \int_G (T_\sigma^{-1} f)(x)\overline{(T_{\sigma^*} G(x)}\, d\mu(x).$$

For all ξ and η in $\widehat{G}$ and all positive integers m, n, k and l with $1 \leq m, n \leq d\xi$ and $1 \leq k, l \leq d_\eta$, let f and g be functions on G sch that

$$f(x) = \xi(x)_{mn}, \quad x \in G$$

and

$$g(x) = \eta(x) =_{kl}, \quad x \in G.$$

Then letting $T_\sigma^{-1} - T_{\sigma^{-1}}$, we get

$$\int_G \xi(x)_{mn}\overline{\eta(x)_{kl}} d\mu(x) = d_\xi d_\eta (\xi(x)\sigma^{-1}(x,\xi))_{mn} \overline{(\eta(x)\sigma^*(x,\eta))_{kl}} d\mu(x).$$

Since $\{\sqrt{d_\xi}\omega_{mn} : 1 \leq m,n \leq d_\omega, \omega \in \widehat{G}\}$ is an orthonormal basis for $L^2(G)$, we have

$$\int_G d_\xi d_\eta (\xi(x)\sigma^{-1}(x,\xi))_{mn}\overline{(\eta(x)\sigma^*(x,\eta))_{kl}}d\mu(x)$$
$$= \begin{cases} 0, & \xi \neq \eta, \\ 0, & m \neq k, \text{ or } n \neq l, \\ \frac{1}{d_\xi}, & \xi = \eta, m = k, 1, n = l. \end{cases}$$

Using the same proof and the fact that for all $f \in L^p(G)$ and $g \in L^{p'}(G)$,

$$(f,g) = (T_{\sigma^{-1}}T_\sigma f, g) = (T_\sigma f, T_{\sigma^{-1*}}g),$$

we get

$$\int_G \frac{1}{d_\xi}\frac{1}{d_\eta}(\xi(x)\sigma(x,\xi))_{mn}\overline{(\eta(x)(\sigma^{-1*}(x,\eta))_{kl}}d\mu(x)$$
$$= \begin{cases} 0, & \xi \neq \eta, \\ 0, & m \neq k, \text{ or } n \neq l, \\ \frac{1}{d_\xi}, & \xi = eta, m = k, n = l. \end{cases} \tag{5.1}$$

Conversely, supoose that there exists a symbol τ on $G \times \widehat{G}$ such that Theorem 5.1 is satisfied. The by Theorem 4.1,

$$T_\sigma T_\tau = T_\tau T_\sigma = I$$

and the proof is complete. □

As a useful corollary, we give a necessary condition for the invertibility of a bounded pseudo-differential operator on G.

Theorem 5.2. *Let σ be a symbol on $G \times \widehat{G}$ such that the corresponding pseudo-differential operator $T_\sigma : L^p(G) \to L^p(G)$ is invertible, where $1 \leq p < \infty$. Then*

$$\int_G \operatorname{tr}(\sigma^{-1}(x,\xi)(\sigma^*(x,\xi))^*\, d\mu(x) = \int_G \operatorname{tr}(\sigma(x,\xi)(\sigma^{*-1}(x,\xi))^*)\, d\mu(x) = d_\xi.$$

Proof Let $\xi \in \widehat{G}$. Then by Theorem 5.1, we get for all positive integers m and n with $1 \leq m, n, \leq d_\xi$,

$$\int_G (\xi(x)\sigma^{-1}(x,\xi))_{mn}\overline{((\xi(x)\sigma^*(x,\xi))_{mn}}d\mu(x)$$
$$= \int_G (\xi(x)\sigma(x,\xi))_{mn}\overline{(\xi(x)\sigma^{-1*}(x,\xi))_{mn}}d\mu(x)$$
$$= \frac{1}{d_\xi}.$$

But for the first integral,

$$\begin{aligned} & \int_G (\xi(x)\sigma^{-1}(x,\xi))_{mn}\overline{(\xi(x)\sigma^*(x,\xi))_{mn}}d\mu(x) \\ + & \int_G (\xi(x)\sigma^{-1}(x,\xi))_{mn}(\xi(x)\sigma^*(x,\xi))^*_{nm}d\mu(x) \\ = & \int_G (\xi(x)\sigma^{-1}(x,\xi))_{mn}((\sigma^*(x,\xi))^*\xi(x)^*)_{nm}d\mu(x) \\ = & \frac{1}{d_\xi} \end{aligned}$$

and so,

$$\begin{aligned} & \sum_{m,n=1}^{d_\xi} \int_G (\xi(x)\sigma^{-1}(x,\xi))_{mn}(\xi(x)^*((\sigma^*(x,\xi))^*)_{nm}d\mu(x) \\ = & \int_G \mathrm{tr}(\sigma^{-1}(x,\xi)(\sigma^*(x,\xi))^*)d\mu(x) \\ = & d_\xi. \end{aligned}$$

Similarly, for the second integral,

$$\int_G \mathrm{tr}(\sigma(x,\xi)(\sigma^{*-1}(x,\xi))^*)\, d\mu(x) = d_\xi$$

and the proof is complete. □

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Majid Jamalpourbirgani
School of Mathematics
Iran University of Science and Technology
Tehran, Tehran Province
Iran
e-mail: m_jamalpour@mathdep.iust.ac.ir

M. W. Wong
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: mwwong@mathstat.yorku.ca