Discrete Analogs of Wigner Transforms and Weyl Transforms

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Abstract. We first introduce the discrete Fourier–Wigner transform and the discrete Wigner transform acting on functions in $L^2(\mathbb{Z})$. We prove that properties of the standard Wigner transform of functions in $L^2(\mathbb{R}^n)$ such as the Moyal identity, the inversion formula, time-frequency marginal conditions and the resolution formula hold for the Wigner transforms of functions in $L^2(\mathbb{Z})$. Using the discrete Wigner transform, we define the discrete Weyl transform corresponding to a suitable symbol on $\mathbb{Z} \times \mathbb{S}^1$. We give a necessary and sufficient condition for the self-adjointness of the discrete Weyl transform. Moreover, we give a necessary and sufficient condition for a discrete Weyl transform to be a Hilbert–Schmidt operator. Then we show how we can reconstruct the symbol from its corresponding Weyl transform. We prove that the product of two Weyl transforms is again a Weyl transform and an explicit formula for the symbol of the product of two Weyl transforms is given. This result gives a necessary and sufficient condition for the Weyl transform to be in the trace class.

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1. Introduction

To put this paper in perspective, we first recall the Wigner transform and the Weyl transform mapping functions in $L^2(\mathbb{R}^n)$ into functions on, respectively, $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n$.

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Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ corresponding to the symbol σ is defined by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi) W(f,g)(x,\xi)\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$, where $W(f,g)$ is the Wigner transform of f and g defined by

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x+\frac{p}{2}\right) \overline{g\left(x-\frac{p}{2}\right)} dp, \quad x,\xi \in \mathbb{R}^n.$$

Closely related to the Wigner transform $W(f,g)$ of f and g in $L^2(\mathbb{R}^n)$ is the Fourier–Wigner transform $V(f,g)$ given by

$$V(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)} dy, \quad q,p \in \mathbb{R}^n.$$

Weyl transforms and Wigner transforms on $\mathbb{R}^n$ have been extensively studied in [4, 13] among others.

Weyl transforms on groups such as the Heisenberg group, the upper half plane and the Poincaré unit disk are investigated in [8, 10, 11, 12]. Closely related to Weyl transforms are pseudo-differential operators on groups. See, for instance, [5, 7, 9, 15].

The strategy that we use to develop the Weyl transform on $\mathbb{Z}$ is to have a look at the case of $\mathbb{R}^n$, where the symbol σ is a function on $\mathbb{R}^n \times \mathbb{R}^n$. Recent works in pseudo-differential operators and Weyl transforms on topological groups G suggest that the correct phase space to work in is $G \times \widehat{G}$, where $\widehat{G}$ is the dual group of G. That the dual group of $\mathbb{R}^n$ is the same as $\mathbb{R}^n$ is the reason why the phase space on which symbols are defined is $\mathbb{R}^n \times \mathbb{R}^n$.

In the case of the group $\mathbb{Z}$ in this paper, the dual group is the unit circle $\mathbb{S}^1$ centered at the origin and the phase space $G \times \widehat{G}$ is then $\mathbb{Z} \times \mathbb{S}^1$.

For $1 \le p < \infty$, the set of all measurable functions F on $\mathbb{Z}$ such that

$$\|F\|^p_{L^p(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |F(n)|^p < \infty$$

is denoted by $L^p(\mathbb{Z})$. We define $L^p(\mathbb{S}^1)$ to be the set of all measurable functions f on the unit circle $\mathbb{S}^1$ with center at the origin for which

$$\|f\|^p_{L^p(\mathbb{S}^1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^p\, d\theta < \infty.$$

We define the Fourier transform $\mathcal{F}_{\mathbb{Z}} F$ of $F \in L^1(\mathbb{Z})$ to be the function on $\mathbb{S}^1$ by

$$(\mathcal{F}_{\mathbb{Z}} F)(\theta) = \sum_{n\in\mathbb{Z}} e^{in\theta} F(n), \quad \theta \in [-\pi,\pi].$$

If f is a suitable function on $\mathbb{S}^1$, then we define the Fourier transform $\mathcal{F}_{\mathbb{S}^1} f$ of f to be the function on $\mathbb{Z}$ by

$$(\mathcal{F}_{\mathbb{S}^1} f)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

Note that $\mathcal{F}_{\mathbb{Z}} : L^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$ is a surjective isomorphism. In fact,

$$\mathcal{F}_{\mathbb{Z}} = \mathcal{F}_{\mathbb{S}^1}^{-1} = \mathcal{F}_{\mathbb{S}^1}^{*}$$

and

$$\|\mathcal{F}_{\mathbb{Z}} F\|_{L^2(\mathbb{S}^1)} = \|F\|_{L^2(\mathbb{Z})}, \quad F \in L^2(\mathbb{Z}).$$

Let H be a suitable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then we define the Fourier transform $\mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} H$ of H to be the function on $\mathbb{Z} \times \mathbb{S}^1$ by

$$(\mathcal{F}_{\mathbb{S}^1 \times \mathbb{Z}} H)(m, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} e^{-im\phi + in\theta} H(\phi, n)\, d\phi, \quad (m, \theta) \in \mathbb{Z} \times \mathbb{S}^1.$$

Similarly, for all suitable functions K on $\mathbb{Z} \times \mathbb{S}^1$, we define the Fourier transform $\mathcal{F}_{\mathbb{Z} \times \mathbb{S}^1} K$ of K to be the function on $\mathbb{S}^1 \times \mathbb{Z}$ by

$$(\mathcal{F}_{\mathbb{Z} \times \mathbb{S}^1} K)(\theta, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n \in \mathbb{Z}} e^{-im\phi + in\theta} K(n, \phi)\, d\phi, \quad (\theta, m) \in \mathbb{S}^1 \times \mathbb{Z}.$$

For $1 \le p < \infty$, we define $L^p(\mathbb{Z} \times \mathbb{S}^1)$ to be the space of all measurable functions h on $\mathbb{Z} \times \mathbb{S}^1$ such that

$$\|h\|_{L^p(\mathbb{Z} \times \mathbb{S}^1)}^p = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} |h(n, \theta)|^p\, d\theta < \infty.$$

In Section 2, we define the Fourier–Wigner transform and the Wigner transform as mappings from $L^2(\mathbb{Z})$ into, respectively, $L^2(\mathbb{Z} \times \mathbb{S}^1)$ and $L^2(\mathbb{S}^1 \times \mathbb{Z})$. Then we show that the discrete Fourier–Wigner transform and the discrete Wigner transform satisfy the Moyal identity. We give an inversion formula to reconstruct a function from its discrete Wigner transform up to a constant factor. Then we give the time and frequency marginal conditions and a convolution theorem for the discrete Wigner transform. The results in this section are analogs of the results for the Wigner transforms on $\mathbb{R}^n$ given in [1, 13]. In Section 3, we use the discrete Wigner transform to define the Weyl transform on $\mathbb{Z}$. A characterization of Hilbert–Schmidt discrete Weyl transforms is also given. The Weyl inversion formula recovering a symbol from the corresponding discrete Weyl transform is given. In Section 4, we present the Weyl calculus giving the symbol of the adjoint of a discrete Weyl transform on $L^2(\mathbb{Z})$ and the symbol of the product of two discrete Weyl transforms. The adjoint formula gives a characterization of self-adjoint discrete Weyl transforms and the product formula gives a characterization of trace class discrete Weyl transforms.

We use $\mathbb{Z}_e$ and $\mathbb{Z}_o$ to denote, respectively, the set of all even integers and the set of all odd integers.

2. Discrete Fourier–Wigner Transforms and Discrete Wigner Transforms

Let $F \in L^2(\mathbb{Z})$. Then for all $(n, \theta) \in \mathbb{Z} \times \mathbb{S}^1$, we define $\rho(n, \theta)F$ to be the function on $\mathbb{Z}$ by

$$(\rho(n,\theta)F)(k) = \begin{cases} e^{i(k+\frac{n}{2})\theta} F(k+n), & n \in \mathbb{Z}_e, \\ e^{i(k+\frac{n-1}{2})\theta} F(k+n), & n \in \mathbb{Z}_o, \end{cases}$$

for all $k \in \mathbb{Z}$. Note that for all $(n, \theta) \in \mathbb{Z} \times \mathbb{S}^1$, $\rho(n, \theta) : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a unitary operator and

$$\rho(n,\theta)^* = \rho(-n, -\theta).$$

For all functions F and G in $L^2(\mathbb{Z})$, we define the Fourier–Wigner transform $V(F, G)$ of F and G to be the function on $\mathbb{Z} \times \mathbb{S}^1$ by

$$V(F,G)(n,\theta) = (\rho(n,\theta)F, G)_{L^2(\mathbb{Z})}, \quad (n,\theta) \in \mathbb{Z} \times \mathbb{S}^1.$$

Therefore for all $(n, \theta) \in \mathbb{Z} \times \mathbb{S}^1$,

$$V(F,G)(n,\theta) = \begin{cases} \sum_{k\in\mathbb{Z}} e^{i(k+\frac{n}{2})\theta} F(k+n)\overline{G(k)}, & n \in \mathbb{Z}_e, \\ \sum_{k\in\mathbb{Z}} e^{i(k+\frac{n-1}{2})\theta} F(k+n)\overline{G(k)}, & n \in \mathbb{Z}_o. \end{cases}$$

By the change of variables from k to m using

$$\begin{cases} m = k + \frac{n}{2}, & n \in \mathbb{Z}_e, \\ m = k + \frac{n-1}{2}, & n \in \mathbb{Z}_o, \end{cases}$$

we get

$$V(F,G)(n,\theta) = \begin{cases} \sum_{m\in\mathbb{Z}} e^{im\theta} F(m+\frac{n}{2})\overline{G(m-\frac{n}{2})}, & n \in \mathbb{Z}_e, \\ \sum_{m\in\mathbb{Z}} e^{im\theta} F(m+\frac{n+1}{2})\overline{G(m-\frac{n-1}{2})}, & n \in \mathbb{Z}_o. \end{cases}$$

In fact, if we let

$$H_n(m) = \begin{cases} F(m+\frac{n}{2})\overline{G(m-\frac{n}{2})}, & n \in \mathbb{Z}_e, \\ F(m+\frac{n+1}{2})\overline{G(m-\frac{n-1}{2})}, & n \in \mathbb{Z}_o. \end{cases}$$

Then

$$V(F,G)(n,\theta) = (\mathcal{F}_{\mathbb{Z}} H_n)(\theta). \tag{2.1}$$

We have the following Moyal identity for the discrete Fourier–Wigner transform.

Theorem 2.1. *Let F_1, F_2, G_1 and G_2 be functions in $L^2(\mathbb{Z})$. Then*

$$\big(V(F_1,G_1), V(F_2,G_2)\big)_{L^2(\mathbb{Z}\times\mathbb{S}^1)} = \big(F_1,F_2\big)_{L^2(\mathbb{Z})}\overline{\big(G_1,G_2\big)_{L^2(\mathbb{Z})}}.$$

Proof For $j = 1, 2$, we let

$$H_{j,n}(m) = \begin{cases} F_j(m + \frac{n}{2})\overline{G_j(m - \frac{n}{2})}, & n \in \mathbb{Z}_e, \\ F_j(m + \frac{n+1}{2})\overline{G_j(m - \frac{n-1}{2})}, & n \in \mathbb{Z}_o. \end{cases}$$

Then by (2.1) and the Parseval identity,

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\pi}^{\pi} V(F_1, G_1)(n, \theta)\overline{V(F_2, G_2)(n, \theta)}\, d\theta \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathcal{F}_{\mathbb{Z}} H_{1,n})(\theta)\overline{(\mathcal{F}_{\mathbb{Z}} H_{2,n})(\theta)}\, d\theta \\ = & \sum_{m\in\mathbb{Z}} H_{1,n}(m)\overline{H_{2,n}(m)}. \end{aligned}$$

Therefore

$$(V(F_1, G_1), V(F_2, G_2))_{L^2(\mathbb{Z}\times\mathbb{S}^1)} = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} H_{1,n}(m)\overline{H_{2,n}(m)}.$$

If $n \in \mathbb{Z}_e$, then we make the change of variables from (m, n) to (k_1, l_1) by $k_1 = m + \frac{n}{2}$ and $l_1 = m - \frac{n}{2}$. If $n \in \mathbb{Z}_o$, then the change of variables from (m, n) to (k_2, l_2) is given by $k_2 = m + \frac{n+1}{2}$ and $l_2 = m - \frac{n-1}{2}$. We get

$$\begin{aligned} & (V(F_1, G_1), V(F_2, G_2))_{L^2(\mathbb{S}^1\times\mathbb{Z})} \\ = & \sum_{\substack{l_1\in\mathbb{Z} \\ k_1+l_1\in\mathbb{Z}_e}}\sum_{k_1\in\mathbb{Z}} F_1(k_1)\overline{G_1(l_1)F_2(k_1)}G_2(l_1) \\ + & \sum_{\substack{l_2\in\mathbb{Z} \\ k_2+l_2\in\mathbb{Z}_o}}\sum_{k_2\in\mathbb{Z}} F_1(k_2)\overline{G_1(l_2)F_2(k_2)}G_2(l_2) \\ = & \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} F_1(k)\overline{G_1(l)F_2(k)}G_2(l) \\ = & \big(F_1, F_2\big)_{L^2(\mathbb{Z})}\overline{\big(G_1, G_2\big)_{L^2(\mathbb{Z})}}. \end{aligned}$$

□

Let F and G be functions in $L^2(\mathbb{Z})$. Then we define the Wigner transform $W(F, G)$ of F and G to be the function on $\mathbb{S}^1 \times \mathbb{Z}$ by

$$W(F, G) = \mathcal{F}_{\mathbb{Z}\times\mathbb{S}^1} V(F, G).$$

Theorem 2.2. *For all* $(\phi, m) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\begin{aligned} & W(F,G)(\phi,m) \\ = & \sum_{n\in\mathbb{Z}_e} e^{in\phi} F\left(m+\frac{n}{2}\right)\overline{G\left(m-\frac{n}{2}\right)} \\ + & \sum_{n\in\mathbb{Z}_o} e^{in\phi} F\left(m+\frac{n+1}{2}\right)\overline{G\left(m-\frac{n-1}{2}\right)}. \end{aligned} \tag{2.2}$$

Proof We begin with the definition of the discrete Wigner transform to the effect that

$$\begin{aligned} & W(F,G)(\theta,m) \\ = & (\mathcal{F}_{\mathbb{Z}\times\mathbb{S}^1} V(F,G))(\theta,m) \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} e^{-im\phi+in\theta} V(F,G)(n,\phi)\,d\phi. \end{aligned}$$

We carry out the sum over $n \in \mathbb{Z}$ by first performing the sum over $n \in \mathbb{Z}_e$ and then over $n \in \mathbb{Z}_o$. Summing over all even integers gives

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n\in\mathbb{Z}_e} e^{-im\phi+in\theta}\sum_{k\in\mathbb{Z}} e^{ik\phi} F\left(k+\frac{n}{2}\right)\overline{G\left(k-\frac{n}{2}\right)}\right)d\phi \\ = & \sum_{n\in\mathbb{Z}_e}\sum_{k\in\mathbb{Z}} e^{in\theta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(m-k)\phi}d\phi\right) F\left(k+\frac{n}{2}\right)\overline{G\left(k-\frac{n}{2}\right)} \\ = & \sum_{n\in\mathbb{Z}_e} e^{in\theta} F\left(m+\frac{n}{2}\right)\overline{G\left(m-\frac{n}{2}\right)} \end{aligned}$$

for all $(\theta, m) \in \mathbb{S}^1 \times \mathbb{Z}$. The sum over $n \in \mathbb{Z}_o$ can be calculated similarly. □

Similarly, we have the Moyal identity for the Wigner transform.

Theorem 2.3. *Let* F_1, F_2, G_1 *and* G_2 *be functions in* $L^2(\mathbb{Z})$. *Then*

$$\big(W(F_1,G_1), W(F_2,G_2)\big)_{L^2(\mathbb{S}^1\times\mathbb{Z})} = \big(F_1,G_1\big)_{L^2(\mathbb{Z})}\overline{\big(F_2,G_2\big)_{L^2(\mathbb{Z})}}.$$

As in the case of Wigner transforms on $\mathbb{R}^n$, the following proposition guarantees that for all $F \in L^2(\mathbb{Z})$, $W(F,F)$ is real.

Proposition 2.4. *Let* F *and* G *be functions in* $L^2(\mathbb{Z})$. *Then*

$$W(F,G) = \overline{W(G,F)}.$$

In particular, $W(F,F)$ *is a real-valued function on* $\mathbb{S}^1 \times \mathbb{Z}$.

Proof For all $(\phi, m) \in \mathbb{S}^1 \times \mathbb{Z}$, we get by (2.2)

$$\begin{aligned} &\overline{W(F,G)(\phi,m)} \\ = &\sum_{n\in\mathbb{Z}_e} e^{-in\phi} G\left(m-\frac{n}{2}\right)\overline{F\left(m+\frac{n}{2}\right)} \\ + &\sum_{n\in\mathbb{Z}_o} e^{-in\phi} G\left(m-\frac{n-1}{2}\right)\overline{F\left(m+\frac{n+1}{2}\right)}. \end{aligned}$$

If we change the index of summation from n to k by $n=-k$, then for all $(\phi, m) \in \mathbb{S}^1 \times \mathbb{Z}$,

$$\begin{aligned} &\overline{W(F,G)(\phi,m)} \\ = &\sum_{k\in\mathbb{Z}_e} e^{ik\phi} G\left(m+\frac{k}{2}\right)\overline{F\left(m-\frac{k}{2}\right)} \\ + &\sum_{k\in\mathbb{Z}_o} e^{ik\phi} G\left(m+\frac{k+1}{2}\right)\overline{F\left(m-\frac{k-1}{2}\right)} \\ = &W(G,F)(\phi,m). \end{aligned}$$

This completes the proof. □

For simplicity, we denote $W(F,F)$ by $W(F)$ for all functions $F \in L^2(\mathbb{Z})$. The following theorem states that we can reconstruct the original function F from its Wigner transform $W(F)$ up to a constant factor.

Theorem 2.5. *Let $F \in L^2(\mathbb{Z})$. Then for all $n \in \mathbb{Z}$,*

$$F(n)\overline{F(0)} = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} W(F)(\phi, \frac{n}{2})\, d\phi, & n \in \mathbb{Z}_e, \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} W(F)(\phi, \frac{n-1}{2})\, d\phi, & n \in \mathbb{Z}_o. \end{cases}$$

Proof By (2.1) and the definition of the Wigner transform, for all m and n in $\mathbb{Z}$, we have

$$H_n(m) = (\mathcal{F}_{\mathbb{S}^1}(W(F)(\cdot, m)))(n).$$

First, we assume that $n \in \mathbb{Z}_e$. Then for all $m \in \mathbb{Z}$, we get

$$F\left(m+\frac{n}{2}\right)\overline{F\left(m-\frac{n}{2}\right)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} W(F)(\phi, m)\, d\phi. \tag{2.3}$$

Now, let $m = \frac{n}{2}$. Then

$$F(n)\overline{F(0)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} W(F)\left(\phi, \frac{n}{2}\right)\, d\phi.$$

Similarly, we obtain $F(n)\overline{F(0)}$, for $n \in \mathbb{Z}_o$ by letting $m = \frac{n-1}{2}$. □

We have the time and frequency marginal conditions for the discrete Wigner transform.

Proposition 2.6. *Let $F \in L^2(\mathbb{Z})$. Then*
(i) *For all $m \in \mathbb{Z}$,*

$$\int_{-\pi}^{\pi} W(F)(\phi, m)\, d\phi = 2\pi |F(m)|^2.$$

(ii) *For all $\phi \in [-\pi, \pi]$,*

$$\sum_{m\in\mathbb{Z}} W(F)(\phi, m) = |\left(\mathcal{F}_{\mathbb{Z}} F\right)(\phi)|^2.$$

Proof Let $n = 0$ in (2.3). Then we get part (i). To prove part (ii), we have for all $\phi \in [-\pi, \pi]$,

$$\begin{aligned} & \sum_{m\in\mathbb{Z}} W(F)(\phi, m) \\ = \; & (\mathcal{F}_{\mathbb{Z}} (W(F)(\phi, \cdot)))(0) \\ = \; & \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_e} e^{in\phi} F\left(m + \frac{n}{2}\right) \overline{F\left(m - \frac{n}{2}\right)} \\ + \; & \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_o} e^{in\phi} F\left(m + \frac{n+1}{2}\right) \overline{F\left(m - \frac{n-1}{2}\right)}. \end{aligned}$$

For all $n \in \mathbb{Z}_e$, we make the change of variables from (m, n) to (k_1, l_1) by $k_1 = m + \frac{n}{2}$ and $l_1 = m - \frac{n}{2}$. Then we get

$$\begin{cases} m = \frac{k_1+l_1}{2}, \\ n = k_1 - l_1, \end{cases} \tag{2.4}$$

and for all $n \in \mathbb{Z}_o$, using the change of variables from from (m, n) to (k_2, l_2) given by $k_2 = m + \frac{n+1}{2}$ and $l_2 = m - \frac{n-1}{2}$, we get

$$\begin{cases} m = \frac{k_2+l_2-1}{2}, \\ n = k_2 - l_2. \end{cases} \tag{2.5}$$

Therefore we get

$$\begin{aligned} & \sum_{m\in\mathbb{Z}} W(F)(\phi, m)\, d\phi \\ = \; & \sum_{\substack{k_1\in\mathbb{Z} \\ k_1+l_1\in\mathbb{Z}_e}} \sum_{l_1\in\mathbb{Z}} e^{i(k_1-l_1)\phi} F(k_1) \overline{F(l_1)} + \sum_{\substack{k_2\in\mathbb{Z} \\ k_2+l_2\in\mathbb{Z}_o}} \sum_{l_2\in\mathbb{Z}} e^{i(k_2-l_2)\phi} F(k_2) \overline{F(l_2)} \\ = \; & \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} e^{i(k-l)\phi} F(k) \overline{F(l)} \\ = \; & |\left(\mathcal{F}_{\mathbb{Z}} F\right)(\phi)|^2 \end{aligned}$$

and the proof is complete. $\square$

Let $T : L^2(\mathbb{Z}\times\mathbb{Z}) \to L^2(\mathbb{Z}\times\mathbb{Z})$ be the twisting operator defined by

$$(TF)(n,m) = \begin{cases} F(m+\frac{n}{2}, m-\frac{n}{2}), & n \in \mathbb{Z}_e, \\ F(m+\frac{n+1}{2}, m-\frac{n-1}{2}), & n \in \mathbb{Z}_o, \end{cases}$$

for all $F \in L^2(\mathbb{Z}\times\mathbb{Z})$ and all $(n,m) \in \mathbb{Z}\times\mathbb{Z}$. In fact, $T : L^2(\mathbb{Z}\times\mathbb{Z}) \to L^2(\mathbb{Z}\times\mathbb{Z})$ is a unitary operator and its inverse T^{-1} is given by

$$(T^{-1}F)(n,m) = \begin{cases} F(n-m, \frac{m+n}{2}), & n+m \in \mathbb{Z}_e, \\ F(n-m, \frac{m+n-1}{2}), & n+m \in \mathbb{Z}_o. \end{cases}$$

Moreover, for all F and G in $L^2(\mathbb{Z})$,

$$W(F,G)(\phi,m) = \left(\mathcal{F}_{1,\mathbb{Z}}T(F\otimes\overline{G})\right)(\phi,m), \quad (\phi,m) \in \mathbb{S}^1\times\mathbb{Z}, \tag{2.6}$$

where $F\otimes\overline{G}$ is the tensor product of F and $\overline{G}$ given by

$$(F\otimes\overline{G})(n,m) = F(n)\overline{G}(m), \quad (n,m) \in \mathbb{Z}\times\mathbb{Z},$$

and $\mathcal{F}_{1,\mathbb{Z}}T(F\otimes\overline{G})$ is the partial Fourier transform of $T(F\otimes\overline{G})$ with respect to the first variable. The following proposition gives the shift-invariance of the Wigner transform and the proof is straightforward.

Proposition 2.7. *Let $F \in L^2(\mathbb{Z})$. For $\theta \in [-\pi,\pi]$ and $k \in \mathbb{Z}$, we define the function G on $\mathbb{Z}$ by*

$$G(n) = e^{in\theta}F(n-k), \quad n \in \mathbb{Z}.$$

Then

$$W(G)(\phi,m) = W(F)(\phi+\theta, m-k), \quad (\phi,m) \in \mathbb{S}^1\times\mathbb{Z}.$$

We can now give a result on the Wigner transform of the product of two functions on $\mathbb{Z}$.

Proposition 2.8. *Let F and G be functions in $L^2(\mathbb{Z})$. Then for all (ϕ,m) in $\mathbb{S}^1\times\mathbb{Z}$,*

$$W(FG)(\phi,m) = \Big(W(F)(\cdot,m) * W(G)(\cdot,m)\Big)(\phi),$$

where $*$ is the convolution on $\mathbb{S}^1$ defined by

$$(f*g)(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi-\theta)g(\theta)\,d\theta$$

for all f and g in $L^2(\mathbb{S}^1)$.

Proof Let $(\phi, m) \in \mathbb{S}^1 \times \mathbb{Z}$. Then

$$\begin{aligned} & (W(F)(\cdot,m) * W(G)(\cdot,m))(\phi) \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi} W(F)(\phi-\theta,m)W(G)(\theta,m)\,d\theta \\ = & \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{\sum_{n_1\in\mathbb{Z}_e} e^{in_1(\phi-\theta)}F\left(m+\frac{n_1}{2}\right)\overline{F\left(m-\frac{n_1}{2}\right)} \\ + & \sum_{n_1\in\mathbb{Z}_o} e^{in_1(\phi-\theta)}F\left(m+\frac{n_1+1}{2}\right)\overline{F\left(m-\frac{n_1-1}{2}\right)}\Big\} \times \\ & \Big\{\sum_{n_2\in\mathbb{Z}_e} e^{in_2\theta}G\left(m+\frac{n_2}{2}\right)\overline{G\left(m-\frac{n_2}{2}\right)} \\ + & \sum_{n_2\in\mathbb{Z}_o} e^{in_2\theta}G\left(m+\frac{n_2+1}{2}\right)\overline{G\left(m-\frac{n_2-1}{2}\right)}\Big\}\,d\theta. \end{aligned}$$

Since

$$\int_{-\pi}^{\pi} e^{-i\theta(n_1-n_2)}\,d\theta = \begin{cases} 0, & n_1 \neq n_2, \\ 2\pi, & n_1 = n_2, \end{cases}$$

it follows that

$$\begin{aligned} & (W(F)(\cdot,m) * W(G)(\cdot,m))(\phi) \\ = & \sum_{n\in\mathbb{Z}_e} e^{in\phi}F\left(m+\frac{n}{2}\right)G\left(m+\frac{n}{2}\right)\overline{F\left(m-\frac{n}{2}\right)G\left(m-\frac{n}{2}\right)} \\ + & \sum_{n\in\mathbb{Z}_o} e^{in\phi}F\left(m+\frac{n+1}{2}\right)G\left(m+\frac{n+1}{2}\right)\times \\ & \qquad \overline{F\left(m-\frac{n-1}{2}\right)G\left(m-\frac{n-1}{2}\right)} \\ = & W(FG)(\phi,m) \end{aligned}$$

for all $(\phi, m) \in \mathbb{S}^1 \times \mathbb{Z}$. □

3. Discrete Weyl Transforms

Let $\sigma \in L^2(\mathbb{Z}\times\mathbb{S}^1)$. Then for all functions F in $L^2(\mathbb{Z})$, we define the Weyl transform $W_\sigma F$ corresponding to the symbol σ by

$$(W_\sigma F, G)_{L^2(\mathbb{Z})} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi}\sigma(m,\phi)W(F,G)(\phi,m)\,d\phi$$

for all $G \in L^2(\mathbb{Z})$. In fact,

$$\begin{aligned}
&(W_\sigma F, G)_{L^2(\mathbb{Z})} \\
= &\ \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}_e} \sigma(m,\phi) e^{in\phi} F\left(m+\frac{n}{2}\right) \overline{G\left(m-\frac{n}{2}\right)}\, d\phi \\
+ &\ \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}_o} \sigma(m,\phi) e^{in\phi} F\left(m+\frac{n+1}{2}\right) \overline{G\left(m-\frac{n-1}{2}\right)}\, d\phi.
\end{aligned}$$

If $n \in \mathbb{Z}_e$, then we make the change of variables from (m,n) to (k_1,l_1) by $k_1 = m + \frac{n}{2}$ and $l_1 = m - \frac{n}{2}$. If $n \in \mathbb{Z}_o$, the change from (m,n) to (k_2,l_2) is effected by $k_2 = m + \frac{n+1}{2}$ and $l_2 = m - \frac{n-1}{2}$. (See (2.4) and (2.5) in this connection.) Therefore we obtain

$$\begin{aligned}
&2\pi\,(W_\sigma F, G)_{L^2(\mathbb{Z})} \\
= &\int_{-\pi}^{\pi} \sum_{\substack{k_1\in\mathbb{Z} \\ k_1+l_1\in\mathbb{Z}_e}} \sum_{l_1\in\mathbb{Z}} e^{i(k_1-l_1)\phi} \sigma\left(\frac{k_1+l_1}{2},\phi\right) F(k_1)\overline{G(l_1)}\, d\phi \\
+ &\int_{-\pi}^{\pi} \sum_{\substack{k_2\in\mathbb{Z} \\ k_2+l_2\in\mathbb{Z}_o}} \sum_{l_2\in\mathbb{Z}} e^{i(k_2-l_2)\phi} \sigma\left(\frac{k_2+l_2-1}{2},\phi\right) F(k_2)\overline{G(l_2)}\, d\phi \\
= &\int_{-\pi}^{\pi} \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} e^{2i(k-l)\phi} \sigma(k,\phi) F(2k-l)\overline{G(l)}\, d\phi \\
+ &\int_{-\pi}^{\pi} \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} e^{i(2(k-l)+1)\phi} \sigma(k,\phi) F(2k+1-l)\overline{G(l)}\, d\phi.
\end{aligned}$$

Therefore for all $l \in \mathbb{Z}$,

$$\begin{aligned}
(W_\sigma F)(l) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} e^{2i(k-l)\phi} \sigma(k,\phi) F(2k-l)\, d\phi \\
&+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} e^{i(2(k-l)+1)\phi} \sigma(k,\phi) F(2k+1-l)\, d\phi.
\end{aligned}$$

By another change of variables, we get

$$(W_\sigma F)(l) = \sum_{m\in\mathbb{Z}} k_\sigma(l,m) F(m),$$

where k_σ is the kernel of W_σ given by

$$k_\sigma(l,m) = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-l)\phi} \sigma(\frac{m+l}{2},\phi)\, d\phi, & m+l \in \mathbb{Z}_e, \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-l)\phi} \sigma(\frac{m+l-1}{2},\phi)\, d\phi, & m+l \in \mathbb{Z}_o. \end{cases} \tag{3.7}$$

Therefore

$$k_\sigma(l,m) = \begin{cases} \left(\mathcal{F}_{2,\mathbb{S}^1}\sigma\right)\left(\frac{m+l}{2}, l-m\right), & m+l \in \mathbb{Z}_e, \\ \left(\mathcal{F}_{2,\mathbb{S}^1}\sigma\right)\left(\frac{m+l-1}{2}, l-m\right), & m+l \in \mathbb{Z}_o. \end{cases} \tag{3.8}$$

where $\mathcal{F}_{2,\mathbb{S}^1}\sigma$ is the Fourier transform on $\mathbb{S}^1$ of σ with respect to the second variable.

The following theorem is an inversion formula for discrete Weyl transforms. It shows how we can reconstruct the symbol from the corresponding Weyl transform. For Weyl transforms on $\mathbb{R}^n$, the corresponding formula and other related formulas can be found in [2, 3, 6].

Theorem 3.1. *Let $\sigma \in L^2(\mathbb{Z}\times\mathbb{S}^1)$ be such that $\rho(n,\theta)W_\sigma$ is a trace class operator for all $(n,\theta)\in\mathbb{Z}\times\mathbb{S}^1$. Then for all $(\theta,n)\in\mathbb{S}^1\times\mathbb{Z}$, we have*

$$(\mathcal{F}_{\mathbb{Z}\times\mathbb{S}^1}\sigma)(\theta,n) = \operatorname{tr}(\rho(n,\theta)W_\sigma).$$

Proof Let $F \in L^2(\mathbb{Z})$. First we assume that $n \in \mathbb{Z}_e$. Then for all $l \in \mathbb{Z}$,

$$\begin{aligned} & \left(\rho(n,\theta)W_\sigma F\right)(l) \\ = {} & e^{i(l+\frac{n}{2})\theta}\left(W_\sigma F\right)(l+n) \\ = {} & e^{i(l+\frac{n}{2})\theta}\sum_{m\in\mathbb{Z}} k_\sigma(l+n,m)F(m), \end{aligned}$$

where k_σ is the kernel of W_σ. So, for all $l \in \mathbb{Z}$,

$$\left(\rho(n,\theta)W_\sigma F\right)(l) = \sum_{m\in\mathbb{Z}} k^\theta(l,m)F(m),$$

where

$$k^\theta(l,m) = e^{i(l+\frac{n}{2})\theta}k_\sigma(l+n,m).$$

Hence

$$\begin{aligned} & \operatorname{tr}(\rho(n,\theta)W_\sigma) \\ = {} & \sum_{l\in\mathbb{Z}} k^\theta(l,l) \\ = {} & \sum_{l\in\mathbb{Z}} e^{i(l+\frac{n}{2})\theta}k_\sigma(l+n,l) \\ = {} & \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} e^{i(l+\frac{n}{2})\theta}\int_{-\pi}^{\pi} e^{-in\phi}\sigma\left(l+\frac{n}{2},\phi\right)\,d\phi. \end{aligned}$$

By the change of variables from l to k by $k = l+\frac{n}{2}$, we get for all $(\theta,n)\in\mathbb{S}^1\times\mathbb{Z}$,

$$\operatorname{tr}(\rho(n,\theta)W_\sigma) = \left(\mathcal{F}_{\mathbb{Z}\times\mathbb{S}^1}\sigma\right)(\theta,n).$$

Similarly, the above formula holds for all $n \in \mathbb{Z}_o$. □

4. Hilbert–Schmidt Discrete Weyl Transforms

The following proposition gives a class of bounded and Hilbert–Schmidt Weyl transforms on $L^2(\mathbb{Z})$.

Proposition 4.1. *Let $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$. Then $W_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a bounded linear operator and*

$$\|W_\sigma\|_* \le \|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)},$$

where $\|\cdot\|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{Z})$. Moreover, W_σ is a Hilbert–Schmidt operator on $L^2(\mathbb{Z})$ and

$$\|W_\sigma\|_{HS(L^2(\mathbb{Z}))} = \|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}.$$

Proof If we define the function $\tilde{\sigma}$ on $\mathbb{S}^1 \times \mathbb{Z}$ by

$$\tilde{\sigma}(\theta, n) = \sigma(n, \theta), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z},$$

then $\tilde{\sigma} \in L^2(\mathbb{S}^1 \times \mathbb{Z})$ and

$$\|\tilde{\sigma}\|_{L^2(\mathbb{S}^1\times\mathbb{Z})} = \|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}.$$

Let F and G be functions in $L^2(\mathbb{Z})$. Then by Schwarz's inequality and the Moyal identity for the Wigner transform, we have

$$\begin{aligned} |(W_\sigma F, G)_{L^2(\mathbb{Z})}| &= |(\tilde{\sigma}, W(G,F))_{L^2(\mathbb{S}^1\times\mathbb{Z})}| \\ &\le \|\tilde{\sigma}\|_{L^2(\mathbb{S}^1\times\mathbb{Z})}\|F\|_{L^2(\mathbb{Z})}\|G\|_{L^2(\mathbb{Z})} \\ &= \|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}\|F\|_{L^2(\mathbb{Z})}\|G\|_{L^2(\mathbb{Z})}. \end{aligned}$$

Hence

$$\|W_\sigma F\|_{L^2(\mathbb{Z})} \le \|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}\|F\|_{L^2(\mathbb{Z})}.$$

Therefore

$$\|W_\sigma\|_* \le \|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}.$$

By (3.8),

$$\begin{aligned} \|W_\sigma\|^2_{HS(L^2(\mathbb{Z}))} &= \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} |k_\sigma(m,l)|^2 \\ &= \sum_{\substack{m\in\mathbb{Z} \\ m+l\in\mathbb{Z}_e}}\sum_{l\in\mathbb{Z}} \left|(\mathcal{F}_{2,\mathbb{S}^1}\sigma)\left(\frac{m+l}{2}, l-m\right)\right|^2 \\ &+ \sum_{\substack{m\in\mathbb{Z} \\ m+l\in\mathbb{Z}_o}}\sum_{l\in\mathbb{Z}} \left|(\mathcal{F}_{1,\mathbb{S}^1}\sigma)\left(\frac{m+l-1}{2}, l-m\right)\right|^2. \end{aligned}$$

By the change of variables and the Parseval identity, we get

$$\begin{aligned} \|W_\sigma\|^2_{HS(L^2(\mathbb{Z}))} &= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left|\left(\mathcal{F}_{2,\mathbb{S}^1}\sigma\right)(n,k)\right|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} |\sigma(n,\phi)|^2\, d\phi \\ &= \|\sigma\|^2_{L^2(\mathbb{Z}\times\mathbb{S}^1)}. \end{aligned}$$

□

A Hilbert–Schmidt operator on $A : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is of the form

$$(AF)(n) = \sum_{m\in\mathbb{Z}} h(n,m)F(m), \quad F \in L^2(\mathbb{Z}),$$

where h is a function in $L^2(\mathbb{Z}\times\mathbb{Z})$. The function h is called the kernel of the Hilbert–Schmidt operator A on $L^2(\mathbb{Z})$. The following theorem states that every Hilbert–Schmidt operator on $L^2(\mathbb{Z})$ is a Weyl transform with symbol in $L^2(\mathbb{Z}\times\mathbb{S}^1)$.

Theorem 4.2. *Let $A : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ be a Hilbert–Schmidt operator. Then there exists a unique symbol σ in $L^2(\mathbb{Z}\times\mathbb{S}^1)$ such that $A = W_\sigma$.*

Proof Let $h \in L^2(\mathbb{Z}\times\mathbb{Z})$ be such that

$$(AF)(n) = \sum_{m\in\mathbb{Z}} h(n,m)F(m), \quad F \in L^2(\mathbb{Z}).$$

Then for all $F, G \in L^2(\mathbb{Z})$,

$$\begin{aligned} (AF,G)_{L^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} h(n,m)F(m)\overline{G(n)} \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \tilde{h}(m,n)\left(F\otimes\overline{G}\right)(m,n) \\ &= (F\otimes\overline{G}, \overline{\tilde{h}})_{L^2(\mathbb{Z}\times\mathbb{Z})}, \end{aligned}$$

where $\tilde{h}$ is the function in $L^2(\mathbb{Z}\times\mathbb{Z})$ such that

$$\tilde{h}(m,n) = h(n,m), \quad (m,n) \in \mathbb{Z}\times\mathbb{Z}.$$

We define the function σ on $\mathbb{Z}\times\mathbb{S}^1$ by

$$\overline{\sigma} = \mathcal{F}_{1,\mathbb{Z}} T\overline{\tilde{h}}^{\sim},$$

where $T : L^2(\mathbb{Z}\times\mathbb{Z}) \to L^2(\mathbb{Z}\times\mathbb{Z})$ is the twisting operator defined earlier. Then $\overline{\tilde{h}} = T^{-1}\mathcal{F}_{1,\mathbb{S}^1}\tilde{\overline{\sigma}}$. Hence we have

$$\begin{aligned} (AF,G)_{L^2(\mathbb{Z})} &= \left(F\otimes\overline{G}, T^{-1}\mathcal{F}_{1,\mathbb{S}^1}\tilde{\overline{\sigma}}\right)_{L^2(\mathbb{Z}\times\mathbb{Z})} \\ &= \left(\mathcal{F}_{1,\mathbb{Z}}T(F\otimes\overline{G}), \tilde{\overline{\sigma}}\right)_{L^2(\mathbb{S}^1\times\mathbb{Z})} \\ &= \left(W(F,G), \tilde{\overline{\sigma}}\right)_{L^2(\mathbb{S}^1\times\mathbb{Z})} \\ &= (W_\sigma F, G)_{L^2(\mathbb{Z})}. \end{aligned}$$

□

5. The Weyl Calculus

The following proposition on the adjoint of a discrete Weyl transform on $L^2(\mathbb{Z})$ follows directly from the definition of the Weyl transform and Proposition 2.4.

Proposition 5.1. *Let $\sigma \in L^2(\mathbb{Z}\times\mathbb{S}^1)$. Then $W_\sigma^* = W_{\overline{\sigma}}$, where $W_\sigma^* : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is the adjoint of $W_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$. In particular, W_σ is self-adjoint if and only if σ is real-valued.*

Proof Let $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$. Then

$$\begin{aligned}(W_\sigma^* F, G)_{L^2(\mathbb{Z})} &= (F, W_\sigma G)_{L^2(\mathbb{Z})} = \overline{(W_\sigma G, F)_{L^2(\mathbb{Z})}} \\ &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi} \overline{\sigma(m,\phi)W(G,F)(\phi,m)}\,d\phi. \qquad (5.1)\end{aligned}$$

By Proposition 2.4,

$$\overline{W(G,F)} = W(F,G),$$

and hence by (5.1),

$$(W_\sigma^* F, G)_{L^2(\mathbb{Z})} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi} \overline{\sigma}(m,\phi)W(F,G)(\phi,m)\,d\phi.$$

So,

$$(W_\sigma^* F, G)_{L^2(\mathbb{Z})} = (W_{\overline{\sigma}} F, G)_{L^2(\mathbb{Z})}$$

and the proof is complete. □

Let $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$. For simplicity, we denote $\mathcal{F}_{\mathbb{Z}\times\mathbb{S}^1}\sigma$ by $\widehat{\sigma}$.

Lemma 5.2. *Let $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$. Then for all $F \in L^2(\mathbb{Z})$,*

$$(W_\sigma F)(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} \widehat{\sigma}(\theta,n)\,(\rho(n,\theta)F)(m)\,d\theta, \quad m \in \mathbb{Z}.$$

Proof Let F and G be in $L^2(\mathbb{Z})$. Then using the adjoint formula,

$$\begin{aligned}(W_\sigma F, G)_{L^2(\mathbb{Z})} &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi}\sigma(m,\phi)W(F,G)(\phi,m)\,d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\hat{\sigma}(\theta,n)V(F,G)(n,\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\hat{\sigma}(\theta,n)\,(\rho(n,\theta)F,G)_{L^2(\mathbb{Z})}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\hat{\sigma}(\theta,n)\,(\rho(n,\theta)F)(m)\overline{G(m)}\,d\theta.\end{aligned}$$

Hence

$$(W_\sigma F)(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\widehat{\sigma}(\theta,n)\,(\rho(n,\theta)F)(m)\,d\theta, \quad m\in\mathbb{Z}.$$

□

Lemma 5.3. *For all* (n,θ) *and* (k,ϕ) *in* $\mathbb{Z}\times\mathbb{S}^1$*, we have*

$$\rho(n,\theta)\rho(k,\phi) = e^{i[(n,\theta);(k,\phi)]}\rho(n+k,\theta+\phi),$$

where

$$[(n,\theta);(k,\phi)] = \begin{cases} \frac{n}{2}\phi - \frac{k}{2}\theta, & n\in\mathbb{Z}_e,\ k\in\mathbb{Z}_e,\\ \frac{n-1}{2}\phi - \frac{k+1}{2}\theta, & n\in\mathbb{Z}_o,\ k\in\mathbb{Z}_o,\\ \frac{n}{2}\phi - \frac{k-1}{2}\theta, & n\in\mathbb{Z}_e,\ k\in\mathbb{Z}_o,\\ \frac{n+1}{2}\phi - \frac{k}{2}\theta, & n\in\mathbb{Z}_o,\ k\in\mathbb{Z}_e. \end{cases}$$

The proof of the lemma is straightforward. Let F and G be suitable functions on $\mathbb{S}^1\times\mathbb{Z}$. Then we define the twisted convolution $F\circledast G$ of F and G to be the function on $\mathbb{S}^1\times\mathbb{Z}$ by

$$\begin{aligned}&(F\circledast G)(\gamma,l)\\ =\ &\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}} e^{i[(l-k,\gamma-\phi);(k,\phi)]}F(\gamma-\phi,l-k)G(\phi,k)\,d\phi, \quad (\gamma,l)\in\mathbb{S}^1\times\mathbb{Z}.\end{aligned}$$

Let $\sigma\in L^2(\mathbb{Z}\times\mathbb{S}^1)$. The following theorem guarantees that the product of two Weyl transforms is still a Weyl transform.

Theorem 5.4. *Let* σ *and* τ *be symbols in* $L^2(\mathbb{Z}\times\mathbb{S}^1)$*. Then*

$$W_\sigma W_\tau = W_\omega,$$

where

$$\hat{\omega} = \hat{\sigma}\circledast\hat{\tau}.$$

Proof Let $F\in L^2(\mathbb{Z})$. Then for all $m\in\mathbb{Z}$, we get by Lemma 5.2

$$\begin{aligned}&(W_\sigma W_\tau F)(m)\\ =\ &\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\widehat{\sigma}(\theta,n)\,(\rho(n,\theta)W_\tau F)(m)\,d\theta\\ =\ &\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\widehat{\sigma}(\theta,n)\widehat{\tau}(\phi,k)\,(\rho(n,\theta)\rho(k,\phi)F)(m)\,d\phi\,d\theta.\end{aligned}$$

Now, by Lemma 5.3, we have

$$\begin{aligned}&(W_\sigma W_\tau F)(m)\\ =\ &\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\widehat{\sigma}(\theta,n)\widehat{\tau}(\phi,k)e^{i[(n,\theta);(k,\phi)]}\times\\ &\qquad(\rho(n+k,\theta+\phi)F)(m)\,d\phi\,d\theta.\end{aligned}$$

Let $l = n + k$ and $\gamma = \theta + \phi$. Then we get

$$\begin{aligned}&(W_\sigma W_\tau F)(m)\\ =\ &\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\widehat{\sigma}(\gamma-\phi, l-k)\widehat{\tau}(\phi,k)\ e^{i[(l-k,\gamma-\phi);(k,\phi)]}\times\\ &\qquad\qquad (\rho(l,\gamma)F)(m)\,d\phi\,d\gamma.\end{aligned}$$

Let $\omega \in L^2(\mathbb{Z}\times\mathbb{S}^1)$ be such that

$$\hat{\omega} = \hat{\sigma} \circledast \hat{\tau}.$$

Then

$$(W_\sigma W_\tau F)(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{l\in\mathbb{Z}}\hat{\omega}(\gamma,l)\,(\rho(l,\gamma)F)(m)\,d\gamma = (W_\omega F)(m)$$

for all $m \in \mathbb{Z}$. □

As an application of the product formula, we give a characterization of trace class discrete Weyl transforms. It is an analog for the discrete Weyl transform of the characterization of trace class Weyl transforms on $\mathbb{R}^n$ in [14]. Let W be the set defined by

$$W = \left\{\mathcal{F}_{\mathbb{S}^1\times\mathbb{Z}}(\hat{\sigma}\circledast\hat{\tau}) :\ \sigma,\tau \in L^2(\mathbb{Z}\times\mathbb{S}^1)\right\}.$$

Let $S_1(L^2(\mathbb{Z}))$ be the space of all trace class operators on $L^2(\mathbb{Z})$. The following theorem gives a characterization of trace class discrete Weyl transforms on $L^2(\mathbb{Z})$.

Theorem 5.5. *Let $\sigma \in L^2(\mathbb{Z}\times\mathbb{S}^1)$. Then $W_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is in $S_1(L^2(\mathbb{Z}))$ if and only if $\sigma \in W$. Moreover, if $\sigma = \mathcal{F}_{\mathbb{S}^1\times\mathbb{Z}}(\hat{\alpha}\circledast\hat{\beta})$ with α and β in $L^2(\mathbb{Z}\times\mathbb{S}^1)$, then*

$$\|W_\sigma\|_{S_1(L^2(\mathbb{Z}))} \le \|\alpha\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}\|\beta\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}.$$

Proof First we assume that $\sigma \in W$. Then

$$\sigma = \mathcal{F}_{\mathbb{S}^1\times\mathbb{Z}}(\hat{\alpha}\circledast\hat{\beta})$$

for some α and β in $L^2(\mathbb{Z}\times\mathbb{S}^1)$. By Theorem 5.4,

$$W_\sigma = W_\alpha W_\beta.$$

Moreover, by Proposition 4.1, W_α and W_β are Hilbert–Schmidt operators. Hence W_σ is the product of two Hilbert–Schmidt operators on $L^2(\mathbb{Z})$ and therefore is in $S_1(L^2(\mathbb{Z}))$. Conversely, assume that $W_\sigma \in S_1(L^2(\mathbb{Z}))$. Then W_σ is the product of two Hilbert–Schmidt operators on $L^2(\mathbb{Z})$. Hence by Theorem 4.2, there exist symbols α and β in $L^2(\mathbb{Z}\times\mathbb{S}^1)$ such that

$$W_\sigma = W_\alpha W_\beta.$$

So, by Theorem 5.4, $\sigma \in W$. □

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