Lecture 26

Theorem (Proved)

Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients on $\mathbb{R}^n$. If $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is Fredholm with zero index, then

$$0 \notin \overline{\{P(\xi) : \xi \in \mathbb{R}^n\}},$$

i.e., there exists a positive constant C such that

$$|P(\xi)| \geq C, \quad \xi \in \mathbb{R}^n.$$

What about the converse?

Theorem Let $P(D)$ be elliptic and such that

$$|P(\xi)| \geq C, \quad \xi \in \mathbb{R}^n.$$

Then $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is Fredholm with zero index.

Proof: We only need to prove that the range $R(P(D))$ of $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is closed in $L^2(\mathbb{R}^n)$. Since $P(D)$ is elliptic, we can find positive constants C' and R such that

$$|P(\xi)| \geq C'(1+|\xi|)^m, \quad |\xi| > R.$$

For $|\xi| \leq R$,

$$\frac{|P(\xi)|}{(1+|\xi|)^m} \geq C''$$

for some $C'' > 0$. i.e.

$$|P(\xi)| \geq C'''(1+|\xi|)^m, \quad \xi \in \mathbb{R}^n.$$

26.2

Let $\tau = \frac{1}{P}$. Then

$$(\partial^\alpha \tau)(\xi) = \sum_{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha} C_{\alpha^{(1)},\ldots,\alpha^{(k)}} \frac{(\partial^{\alpha^{(1)}}P)(\xi)\cdots(\partial^{\alpha^{(k)}}P)(\xi)}{P(\xi)^{k+1}}$$

$\therefore$ $|(\partial^\alpha \tau)(\xi)|$

$$\le \sum_{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha} |C_{\alpha^{(1)},\ldots,\alpha^{(k)}}| \frac{C_{\alpha^{(1)}}\cdots C_{\alpha^{(k)}}(1+|\xi|)^{km-|\alpha|}}{(C'')^{k+1}(1+|\xi|)^{m(k+1)}}$$

$$= \sum_{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha} |C_{\alpha^{(1)},\ldots,\alpha^{(k)}}| \frac{C_{\alpha^{(1)}}\cdots C_{\alpha^{(k)}}}{(C'')^{k+1}}(1+|\xi|)^{-m-|\alpha|}.$$

$\therefore$ $\tau \in S^{-m}$. So, for all $\varphi \in \mathcal{S}$,

$$\|T_\tau \varphi\|_{m,2} \le C_1 \|\varphi\|_2, \quad \varphi \in \mathcal{S}.$$

But

$$\|T_\tau P(D)\varphi\|_{m,2} \le C_1 \|P(D)\varphi\|_2$$

$\varphi \in \mathcal{S}$.

$\therefore$ $\|\varphi\|_{m,2} \le C_1 \|P(D)\varphi\|_2$,

So the range of $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is closed in $L^2(\mathbb{R}^n)$.

Theorem: Let $P(D)=\sum_{|\alpha|\le m} a_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients on $\mathbb{R}^n$. If $P(D): H^{s,2}\to H^{s-m,2}$ is Fredholm with zero index, then $0\notin \overline{\{P(\xi):\xi\in\mathbb{R}^n\}}$. The converse is also true if $P\in S^m$ is elliptic in addition to $0\notin\overline{\{P(\xi):\xi\in\mathbb{R}^n\}}$. 26.3

Proof: Suppose that $P(D): H^{s,2}\to H^{s-m,2}$ is Fredholm with zero index. Then

$$J_{m-s}P(D)J_{s-m}: H^{m,2}\to L^2(\mathbb{R}^n)$$

is Fredholm with zero index. $\therefore$ $P(D): H^{m,2}\to L^2(\mathbb{R}^n)$ is Fredholm with zero index. $\therefore$ $0\notin\overline{\{P(\xi):\xi\in\mathbb{R}^n\}}$.

Now, suppose that $0\notin\overline{\{P(\xi):\xi\in\mathbb{R}^n\}}$ and P is an elliptic polynomial. Then

$$P(D): H^{m,2}\to L^2(\mathbb{R}^n)$$

is Fredholm with zero index. $\therefore$ the result follows.