Theorem: Let $P(D)$ be a linear partial differential operator with constant coefficients of order m on $\mathbb{R}^n$. Then $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is Fredholm with zero index if and only if $0 \notin \{P(\xi) : \xi \in \mathbb{R}^n\}$, i.e., there exists a positive constant C such that

$$|P(\xi)| \geq C, \quad \xi \in \mathbb{R}^n.$$

Lemma 1: Let P be a polynomial on $\mathbb{R}^n$ given by

$$P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

Let $Z(P) = \{\xi \in \mathbb{R}^n : P(\xi) = 0\}$. Then $m(Z(P)) = 0$, where m is the Lebesgue measure on $\mathbb{R}^n$.

Proof: The lemma is certainly true for $n = 1$. Suppose that the lemma is true for all polynomials on $\mathbb{R}^{n-1}$. Write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Any point in $\mathbb{R}^n$ is of the form (ξ', ξ_n), where $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$. Any multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)$ is of the form (α', α_n), where $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$, $\alpha_n \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned} P(\xi) &= P(\xi', \xi_n) \\ &= \sum_{\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \leq m} a_{\alpha', \alpha_n} \xi'^{\alpha'} \xi_n^{\alpha_n} \\ &= \sum_{|\alpha'| \leq m - \alpha_n} \left(\sum_{\alpha_n = 0}^{m} a_{\alpha_n} \xi_n^{\alpha_n} \right) \xi'^{\alpha'} \end{aligned}$$

Let $\chi_{Z(P)}$ be the characteristic function of $Z(P)$.

Then

$$m(Z(P)) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \chi_{Z(P)}(\xi', \xi_n)\, d\xi' d\xi_n$$

$$= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{n-1}} \chi_{Z(P)}(\xi', \xi_n)\, d\xi' \right) d\xi_n$$

$= 0$ by the induction hypothesis.

<u>Lemma 2</u>: Let $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients of order m on $\mathbb{R}^n$. Then

$$\begin{cases} P(D) : H^{m,2} \to L^2(\mathbb{R}^n) \\ P(D)^t : L^2(\mathbb{R}^n) \to H^{-m,2} \end{cases}$$

are both injective.

<u>Proof</u>: Let $u \in H^{m,2}$ be such that $P(D)u = 0$. By the Sobolev embedding theorem, $u \in L^2(\mathbb{R}^n)$. Taking the Fourier transform in distribution sense, we get

$$P(\xi)\hat{u}(\xi) = 0$$

for almost all $\xi \in \mathbb{R}^n$. Since $m(Z(P)) = 0$, $\therefore \hat{u} = 0$ a.e.. $\therefore u = 0$. $\therefore P(D) : H^{m,2} \to L^2(\mathbb{R}^n)$ is injective. Next, let $u \in L^2(\mathbb{R}^n)$ be such that

$$P(D)^t u = 0.$$

Since $P(D)^t : L^2(\mathbb{R}^n) \to H^{-m,2}$ is a bounded linear operator, $\therefore$ for all $v \in H^{m,2}$,

$$0 = (P(D)^t u, v) = (u, P(D)v)$$

$$= (\hat{u}, \bar{P}\hat{v}) = (P\hat{u}, v).$$

$\therefore P\hat{u} = 0 \Rightarrow \hat{u} = 0 \Rightarrow u = 0$. $\therefore P(D)^t : L^2(\mathbb{R}^n) \to H^{-m,2}$ is injective

Proof of Theorem: Suppose that $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is Fredholm with zero index. Then its range is closed in $L^2(\mathbb{R}^n)$. So, there exists a positive constant C such that

$$\|u\|_{m,2} \leq C \|P(D)u\|_2, \quad u \in H^{m,2}$$

$$\therefore \|\varphi\|_2 \leq C \|P(D)\varphi\|_2, \quad \varphi \in \mathcal{S}.$$

$$\therefore 0 \notin \overline{\{P(\xi) : \xi \in \mathbb{R}^n\}}.$$

Conversely, suppose that

$$0 \notin \overline{\{P(\xi) : \xi \in \mathbb{R}^n\}}.$$

Then there exists a positive constant C such that

$$|P(\xi)| \geq C, \quad \xi \in \mathbb{R}^n.$$

Now, we want to prove that the range of $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ is closed in $L^2(\mathbb{R}^n)$. Let $v_k \in$ Range of $P(D): H^{m,2} \to L^2(\mathbb{R}^n)$ such that $v_k \to v$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. For $k = 1, 2, \ldots$, let

$$v_k = P(D)u_k, \quad u_k \in H^{m,2}.$$

Then

$$\|v_k - v_j\|_2 = \|P(D)u_k - P(D)u_j\|_2 = \|P\hat{u}_k - P\hat{u}_j\|_2 \geq C\|\hat{u}_k - \hat{u}_j\|_2$$

$$= C\|u_k - u_j\|_2.$$

$\therefore \{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.

$$C_1 \|u_j - u_k\|_{m,2} \leq \|P(D)(u_j - u_k)\|_2 + \|u_j - u_k\|_2 \to 0$$

as $j, k \to \infty$.

∴ $u_j \to u$ for some u in $H^{m,2}$.

∴ $P(D)u_j \to P(D)u$ in $L^2(\mathbb{R}^n)$ as $j \to \infty$

But $P(D)u_j \to v$ in $L^2(\mathbb{R}^n)$.

∴ $P(D)u = v$. ∴ $v \in$ Range of $P(D) : H^{m,2} \to L^2(\mathbb{R}^n)$.