Lecture 24

Recall: Theorem: Let $\sigma \in S^0$ be such that $T_\sigma: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1<p<\infty$, is Fredholm. Then σ is elliptic.

An Improvement: Theorem: Let $\sigma \in S^m$, $-\infty < m < \infty$. Suppose that $T_\sigma: H^{s,p} \to H^{s-m,p}$ is a Fredholm operator for some $s \in (-\infty,\infty)$. Then T_σ is an elliptic operator.

Remarks: For all $s \in (-\infty,\infty)$, $J_s = T_{\sigma_s}$, where
$$\sigma_s(\xi) = (1+|\xi|^2)^{-s/2}, \quad \xi \in \mathbb{R}^n.$$

Proof of Improved Theorem: Note that
$$\begin{cases} T_\sigma: H^{s,p} \to H^{s-m,p}, \\ J_{-s}: H^{s,p} \to L^p(\mathbb{R}^n), \\ J_{m-s}: H^{s-m,p} \to L^p(\mathbb{R}^n) \end{cases}$$
are bounded linear operators. Let
$$T_\tau = J_{m-s} T_\sigma J_s : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$
Then $\tau \in S^0$. Since J_s is a surjective isometry, $\therefore$ J_s is Fredholm and elliptic. So, T_τ is elliptic.

$\therefore$ $T_\sigma = J_{-s} T_\tau J_{s-m} : H^{s,p} \to H^{s-m,p}$ is elliptic.

$\therefore$ We have "Fredholm $\Rightarrow$ Elliptic". Does "Elliptic $\Rightarrow$ Fredholm"?

Ans: No!

Example: Let $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients on

on $\mathbb{R}^n$ such that $P(D): H^{m,p} \to L^p(\mathbb{R}^n)$ 24.2
is injective and there exists a sequence $\{\xi_k\}_{k=1}^\infty$
in $\mathbb{R}^n$ such that

$$P(\xi_k) \to 0$$

as $k \to \infty$. Then $P(D): H^{m,p} \to L^p(\mathbb{R}^n)$ is not Fredholm. More precisely, its range is not closed in $L^p(\mathbb{R}^n)$.

To see this, let $\varphi \in \mathcal{S}$ be such that $\|\varphi\|_p = 1$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers such that

$$\varepsilon_k^{|\mu|} P^{(\mu)}(\xi_k) \to 0$$

for all μ with $0 < |\mu| \leq m$ as $k \to \infty$. Let $\{\varphi_k\}_{k=1}^\infty$ be the sequence in $\mathcal{S}$ defined by

$$\varphi_k(x) = \varepsilon_k^{+n/p} \varphi(\varepsilon_k x) e^{ix\cdot\xi_k}, \quad x \in \mathbb{R}^n.$$

Then

$$\|\varphi_k\|_p^p = \int_{\mathbb{R}^n} \varepsilon_k^{+n} |\varphi(\varepsilon_k x)|^p dx = \int_{\mathbb{R}^n} |\varphi(y)|^p dy = 1, \quad k \in \mathbb{N}.$$

By Leibniz's formula,

$$(P(D)\varphi_k)(x)$$

$$= \varepsilon_k^{n/p} \sum_{|\mu| \leq m} \frac{1}{\mu!} (P^{(\mu)}(D) e^{ix\cdot\xi_k}) (D^\mu \varphi)(\varepsilon_k x) \varepsilon_k^{|\mu|}$$

$$= \varepsilon_k^{n/p} \sum_{|\mu| \leq m} \frac{1}{\mu!} P^{(\mu)}(\xi_k) e^{ix\cdot\xi_k} (D^\mu \varphi)(\varepsilon_k x) \varepsilon_k^{|\mu|}$$

$$= \varepsilon_k^{n/p} P(\xi_k) e^{ix\cdot\xi_k} \varphi(\varepsilon_k x)$$

$$+ \varepsilon_k^{n/p} \sum_{1 \le |\mu| \le m} \frac{1}{\mu!} P^{(\mu)}(\xi_k) e^{ix\cdot\xi_k} (D^\mu \varphi)(\varepsilon_k x) \varepsilon_k^{|\mu|}.$$

$\therefore$ $\|P(D)\varphi_k\|_p^p$

$$\le \int_{\mathbb{R}^n} \varepsilon_k^n |P(\xi_k)| \, |\varphi(\varepsilon_k x)|^p dx$$

$$+ \int_{\mathbb{R}^n} \varepsilon_k^n \sum_{1 \le |\mu| \le m} \frac{1}{\mu!} |P^{(\mu)}(\xi_k)|^p |(D^\mu \varphi)(\varepsilon_k x)|^p \varepsilon_k^{|\mu| p} dx$$

$$= |P(\xi_k)| + \sum_{1 \le |\mu| \le m} \frac{1}{\mu!} \varepsilon_k^{|\mu| p} |P^{(\mu)}(\xi_k)|^p \int_{\mathbb{R}^n} |(D^\mu \varphi)(y)|^p dy$$

$\longrightarrow 0$

as $k \to \infty$.

So, there cannot be a positive constant C such that $\|u\|_{m,p} \le C \|P(D)u\|_{0,p}$, $u \in H^{m,p}$.

$\therefore$ the range of $P(D): H^{m,p} \to L^p(\mathbb{R}^n)$ is not closed in $L^p(\mathbb{R}^n)$.