Lecture 23

Theorem: Let $\sigma \in S^0$ be such that $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1<p<\infty$, be a Fredholm operator. Then there are positive constants C and R such that

$$|\sigma(x,\xi)| \geq C, \quad |\xi| \geq R,$$

i.e., σ is elliptic.

Proof Since T_σ is Fredholm, we can find a bounded linear operator S on $L^p(\mathbb{R}^n)$ and a compact operator K on $L^p(\mathbb{R}^n)$ such that $ST_\sigma = I - K$ on $L^p(\mathbb{R}^n)$.

Let $M = \{\xi \in \mathbb{R}^n : \exists\, x \in \mathbb{R}^n \ni |\sigma(x,k)| \leq \frac{1}{2\|S\|}\}$.

If M is bounded, then there exists a positive constant R such that

$$\xi \in M \Rightarrow |\xi| \leq R.$$

Then for all ξ with $|\xi| > R$, we have for all $x \in M$,

$$|\sigma(x,\xi)| \geq \frac{1}{2\|S\|}.$$

So, σ is elliptic. Now, suppose that M is not bounded. Then for all $k = 1, 2, \cdots$, there exists a sequence $\{(x_k, \xi_k)\}_{k=1}^{\infty}$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\begin{cases} |\xi_k| \to \infty \text{ as } k \to \infty, \\ |\sigma(x_k, \xi_k)| \leq \frac{1}{2\|S\|}, \quad k = 1, 2, \cdots. \end{cases}$$

For $k = 1, 2, \cdots$, let $\lambda_k = |\xi_k|$. Then

$$R_{\lambda_k,\tau}\left(x_k, \frac{\xi_k}{|\xi_k|}\right)^{-1} T_\sigma R_{\lambda_k,\tau}\left(x_k, \frac{\xi_k}{|\xi_k|}\right) = T_{\sigma_{\lambda_k,\tau}},$$

where

$$\sigma_{\lambda_k,\tau}(x,y) = \sigma(x_k + \lambda_k^{-\tau} x, \xi_k + \lambda_k^{\tau} y), \quad x, y \in \mathbb{R}^n.$$

For all α, β,

$$|(\partial_x^\alpha \partial_y^\beta \sigma_{\lambda_k,\tau})(x,y)| \le C_\beta P_{\alpha\beta}(\sigma)\langle y\rangle^{|\beta|} \lambda_k^{-\tau|\alpha|} \lambda_k^{-(1-2|\alpha|)|\beta|},$$

for all $x, y \in \mathbb{R}^n$. For $k = 1, 2, \cdots$, let

$$\sigma_k^\infty = \sigma_{\lambda_k,\tau}(0,0) = \sigma(x_k, \xi_k).$$

By Taylor's formula, with $\tau \le \frac{1}{2}$

$$|\sigma_{\lambda_k,\tau}(x,y) - \sigma_{\lambda_k,\tau}(0,0)|$$

$$= \left| \sum_{|\gamma|+|\mu|=1} x^\gamma y^\mu \int_0^1 \partial_x^\gamma \partial_y^\mu \sigma_{\lambda_k,\mu}(\theta x, \theta y)\, d\theta \right|$$

$$\le \left| \sum_{|\gamma|+|\mu|=1} |x|^{|\gamma|} |y|^{|\mu|} \int_0^1 C_\mu P_{\gamma,\mu}(\sigma)\langle y\rangle^{|\mu|} \lambda_k^{-(1-2\tau)|\mu|} \lambda_k^{-|\gamma|\tau}\right.$$

$\to 0$ as $k \to \infty$ for all θ uniformly for (x,y) on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$. Let $u \in \mathcal{S}$.

Then

$$(T_{\sigma_{\lambda_k,\tau}} u)(x) - (\sigma_{\lambda_k}^\infty u)(x)$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot y}(\sigma_{\lambda_k,\tau}(x,y) - \sigma_{\lambda_k}^\infty)\hat{u}(y)\, dy$$

for all $x \in \mathbb{R}^n$. Now,

$$\left\{ \begin{array}{l} \text{the integrand} \to 0 \\ \sigma \in S^0 \end{array} \right.$$

$\Rightarrow$ Lebesgue's dominated convergence theorem

$$(T_{\sigma_{\lambda_k,\tau}} u)(x) \to (\sigma_{\lambda_k}^\infty u)(x)$$

for all $x \in \mathbb{R}^n$ as $k \to \infty$.

For all $l \in \mathbb{N}$,

$$\langle x\rangle^{2l}(T_{\sigma_{\lambda_k,\tau}}u)(x)$$

$$= \langle x\rangle^{2l}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot y}\sigma_{\lambda_k,\tau}(x,y)\hat{u}(y)\,dy$$

$$= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\Big(\underbrace{(1-\Delta_y)^l}_{P(D)} e^{ix\cdot y}\Big)\sigma_{\lambda_k,\tau}(x,y)\,\hat{u}(y)\,dy$$

$$\leq (2\pi)^{-n/2}\int_{\mathbb{R}^n} \cancel{e^{ix\cdot y}} \sum_{|\mu|\leq 2l}\left|\frac{1}{\mu!}\left(P^{(\mu)}(D)\hat{u}\right)(y)\right| C_\mu\langle y\rangle^{|\mu|}\,d\mu$$

So there exists a positive C such that

$$|(T_{\sigma_{\lambda_k,\tau}}u)(x)| \leq C\langle x\rangle^{-2l}, \quad x\in\mathbb{R}^n$$

$$\therefore\ |((T_{\sigma_{\lambda_k,\tau}} - \sigma_k^\infty)u)(x)| \leq C\langle x\rangle^{-2l}, \quad x\in\mathbb{R}^n$$

$$\therefore\ T_{\sigma_{\lambda_k,\tau}}u \to \sigma^\infty u \text{ in } L^p(\mathbb{R}^n).$$

$$\therefore\ T_{\sigma_{\lambda_k,\tau}} \quad \text{in } L^p(\mathbb{R}^n). \text{ Then}$$

Let u be a nonzero function in

$$0 \leq \|u\|_p = \left\|R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)u\right\|_p$$

$$= \left\|(ST_\sigma - K)R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)u\right\|_p$$

$$\leq \left\|ST_\sigma R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)u\right\|_p + \left\|K R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)u\right\|_p$$

$$\leq \|S\|\left\|R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)^{-1}T_\sigma R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)u\right\|_p + \left\|K\left(R_{\lambda_k,\tau}\left(x_k,\frac{\xi_k}{|\xi_k|}\right)u\right\|_p \to 0$$

$$\therefore\ \|u\|_p \leq \|S\|\,|\sigma^\infty|\,\|u\|_p$$

$$\therefore\ 1 \leq \|S\|\,|\sigma_k^\infty|$$

$$\therefore\ \frac{1}{\|S\|} \leq |\sigma^\infty| \leq \frac{1}{2\|S\|}.$$