Definition: Let $\lambda > 0$, $\tau \geq 0$, $x_0, \xi_0 \in \mathbb{R}^n$. Then we define $R_{\lambda,\tau}(x_0,\xi_0) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1<p<\infty$, by

$$(R_{\lambda,\tau}(x_0,\xi_0)u)(x) = \lambda^{\tau n/p} e^{i\lambda x\cdot\xi_0} u(\lambda^{\tau}(x-x_0)), \quad x \in \mathbb{R}^n,$$

for all $u \in L^p(\mathbb{R}^n)$.

Proposition: $R_{\lambda,\tau}(x_0,\xi_0) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a a surjective isometry.

Proof: (Isometry) For all $u \in L^p(\mathbb{R}^n)$,

$$\| R_{\lambda,\tau}(x_0,\xi_0)u \|_p^p = \int_{\mathbb{R}^n} \lambda^{\tau n} |u(\lambda^{\tau}(x-x_0))|^p dx \qquad y = \lambda^{\tau}(x-x_0)$$

$$= \int_{\mathbb{R}^n} |u(y)|^p dy = \|u\|_p.$$

(Surjectivity)

Let $v \in L^p(\mathbb{R}^n)$. Let

$$u(x) = \lambda^{-\tau n/p} e^{i\lambda(x_0+\lambda^{-\tau}x)\cdot\xi_0} v((x_0+\lambda^{-\tau}x)), \quad x \in \mathbb{R}^n.$$

Then

$$(R_{\lambda,\tau}(x_0,\xi_0)u)(x) = \lambda^{\tau n/p} e^{i\lambda x\cdot\xi_0} u(\lambda^{\tau}(x-x_0))$$

$$= \lambda^{\tau n/p} e^{i\lambda x\cdot\xi_0} \lambda^{-\tau n/p} e^{i\lambda(x_0+x-x_0)\cdot\xi_0} v(x)$$

$$= v(x), \quad x \in \mathbb{R}^n.$$

Proposition: For all $u \in L^p(\mathbb{R}^n)$, $v \in L^{p'}(\mathbb{R}^n)$ for $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$,

$$(R_{\lambda,\tau}(x_0,\xi_0)u, v) \to 0$$

as $\lambda \to \infty$.

Proof: Let $u, v \in C_0^\infty(\mathbb{R}^n)$. Then

$$|(R_{\lambda,\tau}(x_0,\xi_0)u, v)| \le \lambda^{\tau n/p} \int_{\mathbb{R}^n} |u(\lambda^\tau(x-x_0))||v(x)|\,dx$$

$$(y = \lambda^\tau(x-x_0))$$

$$= \lambda^{\tau n/p} \int_{\mathbb{R}^n} |u(y)||v(x_0+\lambda^{-\tau}y)|\,dy\,\lambda^{-n\tau}$$

$$= \lambda^{-n\tau/p'} \int_{\mathbb{R}^n} |u(y)||v(x_0+\lambda^{-\tau}y)|\,dy$$

$\to 0$ as $\lambda \to \infty$.

By density, let $\{\varphi_j\}_{j=1}^\infty$ and $\{\psi_j\}_{j=1}^\infty$ be sequences in $C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \varphi_j \to u & \text{in } L^p(\mathbb{R}^n) \\ \psi_j \to v & \text{in } L^{p'}(\mathbb{R}^n) \end{cases}$$

as $j \to \infty$. Then for every $\varepsilon > 0$, there exists a positive integer J such that

$$|(R_{\lambda,\tau}(x_0,\xi_0)u, v) - (R_{\lambda,\tau}(x_0,\xi_0)\varphi_j, \psi_j)| < \varepsilon/2$$

for $j \ge J$.

Now,

$$|(R_{\lambda,\tau}(x_0,\xi_0)u,v)| \le |(R_{\lambda,\tau}(x_0,\xi_0)u,v) - R_{\lambda,\tau}(x_0,\xi_0)\varphi_J, \psi_J)| + |(R_{\lambda,\tau}(x_0,\xi_0)\varphi_J, \psi_J)|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \text{whenever } \lambda > \lambda_0 \text{ for some } \lambda_0.$$

Proposition

Let $\sigma \in S^m$. Then

$$R_{\lambda,\tau}(x_0,\xi_0)^{-1} T_\sigma R_{\lambda,\tau}(x_0,\xi_0) = T_{\sigma,\lambda},$$

where $\sigma_{\sigma,\lambda}(x,\eta) = \sigma(x_0 + \lambda^{-\tau}x, \lambda\xi_0 + \lambda^\tau\eta)$, $x, \eta \in \mathbb{R}^n$.

Moreover, if $\sigma \in S^0$, and $\xi_0 \ne 0$, then for all α, β

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma_{\lambda,\tau})(x,\eta)| \le C_\beta P_{\alpha\beta}(\sigma) \frac{\langle\eta\rangle^\beta}{|\xi_0|^{|\beta|}} \lambda^{-\tau|\alpha|} \lambda^{-(1-2\tau)|\beta|}, \quad x, \eta \in \mathbb{R}^n.$$

Lemma (Peetre's Inequality)

For all $t \in (-\infty, \infty)$ and all $x, y \in \mathbb{R}^n$,

$$\left(\frac{1+|x|^2}{1+|y|^2}\right)^t \leq 2^{|t|}\left(1+|x-y|^2\right)^{|t|}.$$

Proof: The inequality is obviously true for $t = 0$. Now for all $y, z \in \mathbb{R}^n$,

$$\begin{aligned} 1+|y-z|^2 &= 1+(y-z)\cdot(y-z) \\ &= 1+|y|^2-2y\cdot z+|z|^2 \\ &\leq 1+|y|^2+|y|^2+|z|^2+|z|^2 \\ &= 1+2|y|^2+2|z|^2 \\ &= 2(1+|y|^2)(1+|z|^2). \end{aligned}$$

Let $x = y - z \therefore z = y - x$. Then

$$1+|x|^2 \leq 2(1+|y|^2)(1+|x-y|^2).$$

$$\therefore \left(\frac{1+|x|^2}{1+|y|^2}\right)^t \leq 2^t\left(1+|x-y|^2\right)^t \text{ if } t > 0.$$

If $t < 0$, then $-t > 0$.

$$\therefore \left(\frac{1+|x|^2}{1+|y|^2}\right)^{-t} \leq 2^{-t}\left(1+|x-y|\right)^{-t}.$$

$$\therefore \left(\frac{1+|x|^2}{1+|y|^2}\right)^{|t|} \leq 2^{|t|}\left(1+|x-y|\right)^{|t|}.$$