Lecture 20

Definition Let $\lambda > 0$, $\tau \geq 0$, $x_0, \xi_0 \in \mathbb{R}^n$. Then we define $R_{\lambda,\tau}(x_0,\xi_0) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1<p<\infty$, by

$$(R_{\lambda,\tau}(x_0,\xi_0)u)(x) = \lambda^{\tau n/p} e^{i\lambda x\cdot\xi_0} u(\lambda^\tau (x-x_0)), \quad x \in \mathbb{R}^n,$$

for all $u \in L^p(\mathbb{R}^n)$.

Proposition: $R_{\lambda,\tau}(x_0,\xi_0) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a surjective isometry.

Proof (Isometry)

For all $u \in L^p(\mathbb{R}^n)$

$$\| R_{\lambda,\tau}(x_0,\xi_0)u\|_p^p = \left(\int_{\mathbb{R}^n} (e^{\tau n} |u(\lambda^n(x-x_0))|^p dx\right)^p$$

Let $y = \lambda^\tau(x-x_0)$

Then $\| R_{\lambda,\tau}(x_0,\xi_0)u\|_p^p = \left(\int_{\mathbb{R}^n} |u(y)|^p dy\right)^{\frac{1}{p}} = \|u\|_p^p$

(Surjectivity) Let $v \in L^p(\mathbb{R}^n)$. Let

$$u(x) = \lambda^{-\tau n/p} e^{i\lambda(x_0+\lambda^{-\tau}x)\cdot\xi_0} v(x_0+\lambda^{-1}x), \quad x \in \mathbb{R}^n.$$

Then $u \in L^p(\mathbb{R}^n)$ and

$$\begin{aligned}
(R_{\lambda,\tau}(x_0,\xi_0)u)(x) &= e^{i\lambda(x\cdot\xi_0)} u(\lambda^\tau(x-x_0)) \\
&= e^{i\lambda x\cdot\xi_0} u(\lambda^\tau(x-x_0)) \\
&= e^{i\lambda x\cdot\xi_0} e^{i\lambda(x_0+x-x_0)\cdot\xi_0} v(x_0+x-x_0) \\
&= v(x), \quad x \in \mathbb{R}^n.
\end{aligned}$$

<u>Proposition</u> For all $u \in L^p(\mathbb{R}^n)$ and $v \in L^{p'}(\mathbb{R}^n)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

$$(R_{\lambda,\tau}(x_0, \xi_0) u, v) \to 0$$

as $\lambda \to \infty$.

<u>Proof</u>: Let $u, v \in C_0^\infty(\mathbb{R}^n)$. Then

$$|(R_{\lambda,\tau}(x_0,\xi_0) u, v)| \leq \lambda^{\tau n/p} \int_{\mathbb{R}^n} |u(\lambda^\tau(x - x_0))| |v(x)| dx$$

$$\leq \lambda^{\tau n/p} \int_{\mathbb{R}^n} |u(\lambda^\tau(x - x_0))| |v(x)| dx$$ Let $y = \lambda^{+\tau}(x - x_0)$.

$$= \lambda^{-\tau n/p'} \int_{\mathbb{R}^n} |u(y)| |v(x_0 + \lambda^{-\tau} y)| dy$$

$\to 0$ as $\lambda \to \infty$.

Let $u \in L^p(\mathbb{R}^n)$, $v \in L^{p'}(\mathbb{R}^n)$.

Then there exist sequences $\{\varphi_j\}_{j=1}^\infty$ and $\{\psi_j\}_{j=1}^\infty$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \varphi_j \to u \text{ in } L^p(\mathbb{R}^n) \\ \psi_j \to v \text{ in } L^{p'}(\mathbb{R}^n) \end{cases}$$

as $j \to \infty$.

Therefore

$$|(R_{\lambda,\tau}(x_0,\xi_0) u, v)| = \lim_{j\to\infty} (R_{\lambda,\tau}(x_0,\xi_0) \varphi_j, \psi_j)|$$

$$\to 0.$$

Proposition

Let $\sigma \in S^m$. Then

$$R_{\lambda,\tau}(x_0,\xi_0)^{-1} T_\sigma R_{\lambda,\tau}(x_0,\xi_0) = T_{\sigma_{\lambda,\tau}},$$

where

$$\sigma_{\lambda,\tau}(x,y) = \sigma(x_0 + \lambda^{-\tau}x, \lambda\xi_0 + \lambda^{\tau}y), \quad x, y \in \mathbb{R}^n.$$

Moreover, if $\sigma \in S^0$, and $\xi_0 \neq 0$, then for all for all multiindices α, β, there exists a $C_\beta > 0$ such that

$$|(\partial_x^\alpha \partial_y^\beta \sigma)(x,y)| \leq C_\beta P_{\alpha\beta}(\sigma) \frac{\langle y \rangle^{|\beta|}}{|\xi_0|^{|\beta|}} |\lambda|^{-\tau|\alpha|} |\lambda|^{-(1-2\tau)|\beta|},$$

$$x, y \in \mathbb{R}^n.$$