Lecture 8 (Chapter 17)

Let X be a complex Banach space. Let $\underset{\underset{X}{\cap\ \text{dense subspace}}}{A} \to X$ be a closed linear operator. Let $f \in X$.

Consider
$$\begin{cases} u'(t) = A(u(t)), \ t > 0, \\ u(0) = f, \end{cases}$$
where $u : [0, \infty) \to X$ and
$$u'(t) = \lim_{h \to 0+} \frac{u(t+h) - u(t)}{h}$$
if the limit exists.

Questions: Does a global solution $u : [0, \infty) \to X$ exist? If yes, is it unique?

Intuition: Is the solution $u : [0, \infty) \to X$ is given formally by $u(t) = e^{tA} f$? (To be justified)

Main Question: What is e^{tA} ?

Definition: Let $\{T(t) : t \geq 0\}$ be a family of bounded linear operators on X. Suppose that

- $T(0) = I$, where I is the identity operator on X,
- $T(s)T(t) = T(s+t)$, $s \geq 0$, $t \geq 0$,
- $T(t)x \to x$ in X as $t \to 0+$.

Then we call $\{T(t) : t \geq 0\}$ a one-parameter semigroup on X.

Let $\{T(t) : t \geq 0\}$ be a one-parameter semigroup on X. Define $\mathcal{D}(A)$ by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t \to 0+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

<u>Proposition</u>: $\mathcal{D}(A)$ is dense in X.

<u>Proof</u>: Let W be the subspace of X given by

$$W = \left\{ x_s = \frac{1}{s}\int_0^s T(\delta)x\,d\delta,\ x \in X,\ \delta > 0 \right\}.$$

W is dense in X because for all x in X, let

$$x_s = \frac{1}{s}\int_0^s T(\delta)x\,d\delta.$$

Then

$$x_s - x = \frac{1}{s}\int_0^s \left(T(\delta)x - x\right)d\delta.$$

For all positive numbers ε,

$$\| x_s - x \| \leq \frac{1}{s}\int_0^s \| T(\delta)x - x \|\,d\delta.$$

$\therefore \exists \delta_1 > 0$ such that

$$0 < \delta < \delta_1 \Longrightarrow \| T(\delta)x - x \| < \varepsilon.$$

Now $0 < s < \delta_1$

$$\Longrightarrow \| x_s - x \| \leq \frac{1}{s}\int_0^s \varepsilon\,d\delta = \varepsilon.$$

$\therefore W$ is dense in X. For $t > 0$, let

$$A_t = \frac{T(t) - I}{t}.$$

Let $s, t < 0$ with $t < s$. Then

$$A_t x_s = \frac{T(t)\int_0^s T(\delta)x\,d\delta - \int_0^s T(\delta)x\,d\delta}{st}$$

$$= \frac{\int_0^s T(t+\delta)x\,d\delta - \int_0^s T(\delta)x\,d\delta}{st} \qquad (t+\delta = y)$$

$$= \frac{\int_t^{s+t} T(y)x\,dy - \int_0^s T(\delta)x\,d\delta}{st}$$

By symmetry,

$$A_s x_t = \frac{\int_s^{s+t} T(\delta)x\,d\delta - \int_0^t T(\delta)x\,d\delta}{st}.$$

Let $t \to 0+$. Then

$$A_t x_s \to A_s x \text{ in } X.$$

$\therefore$ $x_s \in \mathcal{D}(A)$. $\therefore$ $\mathcal{D}(A)$ is dense in X.

<u>Definition</u> Let $A : \mathcal{D}(A) \to X$ be defined by

$$\underset{X}{\overset{\text{dense subspace}}{\cap}}$$

$$Ax = \lim_{t \to 0+} \frac{T(t)x - x}{t}, \quad x \in \mathcal{D}(A),$$

where the limit is to take place in X. Then we call A the <u>infinitesimal generator</u> of $\{T(t) : t \geq 0\}$

<u>Proposition</u>: The infinitesimal generator A of a one-parameter semigroup $\{T(t) : t \geq 0\}$ on a complex Banach space X is a closed linear operator.

<u>Lemma</u>: Let A be the infinitesimal generator of a one-parameter semigroup $\{T(t) : t \geq 0\}$ on a complex Banach space X. Then for all $t \in [0, \infty)$ and all $x \in \mathcal{D}(A)$,

$$T(t)Ax = AT(t)x.$$

<u>Proof</u>:

$$T(t)A_h x = T(t)\frac{T(h) - I}{h}x = \frac{T(t+h) - T(t)}{h}x$$

$$= \frac{T(h) - I}{h}T(t)x \to AT(t)x.$$

But $T(t)A_h x \to T(t)Ax$.

$\therefore$ $T(t)Ax = AT(t)x$.