Lecture 7

Recall

Theorem: Let $\sigma \in S^{2m}$, $m > 0$, be an elliptic symbol such that there exists a positive constant C such that

$$\mathrm{Re}\,(T_\sigma \varphi, \varphi) \geq C \|\varphi\|^2_{m,2}, \quad \varphi \in \mathcal{S}.$$

Then for all f in $L^2(\mathbb{R}^n)$, the pseudo-differential equation $T_\sigma u = f$ on $\mathbb{R}^n$ has a strong solution u in $L^2(\mathbb{R}^n)$.

Remark: A better theorem should be one with conditions imposed on σ, not on T_σ.

Theorem: Let $\sigma \in S^{2m}$, $m \geq \frac{1}{2}$, be a strongly elliptic symbol. Then there exists a real number λ_0 such that for all f in $L^2(\mathbb{R}^n)$ and all $\lambda \geq \lambda_0$, the pseudo-differential equation $(T_\sigma + \lambda I)u = f$ on $\mathbb{R}^n$, where I is the identity operator on $L^2(\mathbb{R}^n)$, has a unique strong solution in $L^2(\mathbb{R}^n)$.

Proof: By Gårding's inequality, we can find a positive constant A and a real constant C_s for all $s \geq \frac{1}{2}$ such that

$$\mathrm{Re}\,(T_\sigma \varphi, \varphi) \geq A \|\varphi\|^2_{m,2} - C_s \|\varphi\|^2_{m-s,2}, \quad \varphi \in \mathcal{S}.$$

So there exists a real constant λ_0 such that

$$\mathrm{Re}\,(T_\sigma \varphi, \varphi) \geq A \|\varphi\|^2_{m,2} - \lambda_0 \|\varphi\|^2_{0,2}, \quad \varphi \in \mathcal{S}.$$

Then for $\lambda \geq \lambda_0$,

$$\mathrm{Re}\,((T_\sigma + \lambda I)\varphi, \varphi) \geq A \|\varphi\|^2_{m,2} + (\lambda - \lambda_0)\|\varphi\|^2_{0,2} \geq A \|\varphi\|^2_{m,2}, \quad \varphi \in \mathcal{S}.$$

So $(T_\sigma + \lambda I)u = f$ on $\mathbb{R}^n$ has a unique strong solution u in $L^2(\mathbb{R}^n)$.

<u>Theorem</u>: Let $\sigma \in S^m$, $m > 0$, be an elliptic symbol such that σ is independent of x in $\mathbb{R}^n$ and

$$\sigma(\xi) \neq 0, \quad \xi \in \mathbb{R}^n.$$

Then for every function f in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, the pseudo-differential equation $T_\sigma u = f$ on $\mathbb{R}^n$ has a unique strong solution u in $L^p(\mathbb{R}^n)$.

<u>Proof</u>: By Exercise 18.3, it is sufficient to prove that there exists a positive Constant C such that

$$\|\varphi\|_{p'} \leq C \| T_\sigma^* \varphi \|_{p'}, \quad \varphi \in \mathcal{S}.$$

Let $\tau = \frac{1}{\bar{\sigma}}$. Then for all multi-indices α,

$$\partial^\alpha \tau = \sum C_{\alpha^{(1)}, \dots, \alpha^{(k)}} \frac{(\partial^{\alpha^{(1)}} \bar{\sigma}) \cdots (\partial^{\alpha^{(k)}} \bar{\sigma})}{\bar{\sigma}^{k+1}},$$

where $C_{\alpha^{(1)}, \dots, \alpha^{(k)}}$ is a constant depending on $\alpha^{(1)}, \dots, \alpha^{(k)}$ and the sum is taken over all multi-indices $\alpha^{(1)}, \dots, \alpha^{(k)}$ partitioning α. So we can find positive constants $\dots, C^{\alpha^{(k)}}$ such that
$C^{\alpha^{(1)}}$

$$|(\partial^\alpha \tau)(\xi)| \leq \sum |C_{\alpha^{(1)}, \dots, \alpha^{(k)}}| \frac{C^{\alpha^{(1)}} \cdots C^{\alpha^{(k)}} (1+|\xi|)^{km-|\alpha|}}{|\sigma(\xi)|^{k+1}},$$

Since σ is elliptic, there exist positive constants $\{\xi \in \mathbb{R}^n$:
C and R with

$$|\sigma(\xi)| \geq C(1+|\xi|)^m, \quad |\xi| \geq R.$$

Now,

$$\frac{|\sigma|}{(1+|\ |)^m}$$ is a continuous and positive function on $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$. So, there exists a positive

constant δ such that
$$|\sigma(\xi)| \geq \delta(1+|\xi|)^m, \quad |\xi| \leq R.$$
$\therefore$ there exists a positive constant C' such that
$$|\sigma(\xi)| \geq C'(1+|\xi|)^m, \quad \xi \in \mathbb{R}^n.$$
$\therefore$ there exists a positive constant C'' such that
$$|(\partial^\alpha \tau)(\xi)| \leq C''(1+|\xi|)^{-m-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$
$\therefore \tau \in S^{-m}$. By the L^p-boundedness theorem,
$$\|\varphi\|_{p'} = \|T_\tau T_{\bar\sigma}\varphi\|_{p'} \leq \|T_{\bar\sigma}\varphi\|_{-m,p} \leq \|T_{\bar\sigma}\varphi\|_{p'}.$$
For uniqueness, let u and v be strong solutions in $L^p(\mathbb{R}^n)$. Let $\omega = u - v$. Then $\omega \in H^{m,p}$ and $T_\sigma \omega = 0$ on $\mathbb{R}^n$. $\therefore$
$$\sigma\hat{\omega} = 0$$
in $\mathcal{S}$. So, for all $\varphi \in \mathcal{S}$,
$$(\sigma\hat{\omega})(\varphi) = 0,$$
i.e.,
$$\hat{\omega}(\sigma\varphi) = 0.$$
But for all $\psi \in \mathcal{S}$, there exists a function φ in $\mathcal{S}$ such that $\psi = \sigma\varphi$. (18.5). So,
$$\hat{\omega}(\psi) = 0, \quad \psi \in \mathcal{S}.$$
$\therefore \hat{\omega} = 0$. $\therefore \omega = 0$. $\therefore u = v$.