Lecture 5

Theorem: Let $\sigma \in S^{2m}$, $m>0$, be an elliptic symbol such that there exists a positive constant C for which

$$Re(T_\sigma\varphi, \varphi) \geq C\|\varphi\|_{m,2}^2, \quad \varphi \in \mathcal{S}.$$

Then for every $f \in L^2(\mathbb{R}^n)$, there exists a unique strong solution u in $L^2(\mathbb{R}^n)$ of $T_\sigma u = f$ on $\mathbb{R}^n$, i.e.,

$$u \in \mathcal{D}(T_{\sigma,0}), \quad T_{\sigma,0}u = f.$$

Lemma Under the same hypotheses of the Theorem, there exists a positive constant C such that

$$Re(T_\sigma u, u) \geq C\|u\|_{m,2}^2, \quad u \in H^{m,2}.$$

Proof: Let $u \in H^{m,2}$. Then there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ such that $\varphi_k \to u$ in $H^{m,2}$. By the L^2-boundedness of pseudo-differential operators, there exists a positive constant C' such that

$$|(T_\sigma\varphi_k, \varphi_k) - (T_\sigma u, u)|$$

$$= |(T_\sigma\varphi_k, \varphi_k) - (T_\sigma u, \varphi_k) + (T_\sigma u, \varphi_k) - (T_\sigma u, u)|$$

$$\leq |(T_\sigma(\varphi_k - u), \varphi_k)| + |(T_\sigma u, \varphi_k - u)|$$

$$\leq C'(\|\varphi_k - u\|_{m,2}\|\varphi_k\|_{m,2} + \|u\|_{m,2}\|\varphi_k - u\|_{m,2}) \to 0 \text{ as } k \to \infty.$$

$$\therefore Re(T_\sigma u, u) = \lim_{k\to\infty} Re(T_\sigma\varphi_k, \varphi_k) \geq C \lim_{k\to\infty}\|\varphi_k\|_{m,2}^2 = C\|u\|_{m,2}^2$$

$u \in H^{m,2}$.

Theorem (Lax-Milgram Theorem) (Lemma) Let X be a complete and separable Hilbert space with inner product $(\,,\,)_X$ and norm $\|\ \|_X$. Let B be a bilinear mapping on X such that there exist positive constants C_1 and C_2 for which

$$|B(x,y)| \leq C_1\|x\|_X\|y\|_X, \quad x, y \in X,$$

$$|B(x,x)| \geq C_2\|x\|_X^2, \quad x \in X.$$

Then for every bounded linear functional on X, there exists a unique $y \in X$ such that

$$f(x) = B(x, y), \quad x \in X.$$

<u>Proof</u>: Let $y \in X$. Then $B(\cdot, y)$ is a bounded linear functional on X. By the Riesz representation theorem, there exists a unique $z(y)$ in X with

$$B(x, y) = (x, z(y)), \quad x \in X.$$

Now

$$X \ni y \mapsto z(y) \in X$$

is linear. Indeed let $y_1, y_2 \in X$ and $c_1, c_2 \in \mathbb{C}$. Then

$$\begin{aligned}
(x, z(c_1 y_1 + c_2 y_2)) = B(x, c_1 y_1 + c_2 y_2) &= \bar{c}_1 B(x, y_1) + \bar{c}_2 B(x, y_2) \\
&= \bar{c}_1 (x, z(y_1))_X + \bar{c}_2 (x, z(y_2))_X \\
&= \bar{c}_1 B(x, y_1) + \bar{c}_2 B(x, y_2). \\
&= (x, c_1 z(y_1))_X + (x, c_2 z(y_2))_X
\end{aligned}$$

Let M be a subspace of X be defined by

$$M = \{ z(y) \in X, \ y \in X \}.$$

Then M is a closed subspace of X. To see this, let $\{z(y_k)\}_{k=1}^{\infty}$ be a sequence in M such that

$$z(y_k) \to z$$

as $k \to \infty$.

Then for $j, k = 1, 2, \cdots,$

$$B(x, y_j - y_k) = (x, z(y_j) - z(y_k))_X, \quad x \in X.$$

So

$$\begin{aligned}
|B(y_j - y_k, y_j - y_k)| &\geq C_2 \| y_j - y_k \|_X^2 \\
\| \\
|(y_j - y_k, z(y_j) - z(y_k))_X| \\
\wedge\!| \\
\| y_j - y_k \|_X \, \| z(y_j) - z(y_k) \|_X .
\end{aligned}$$

But $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X. $\therefore y_k \to y$ for some y in X as $k \to \infty$. $\therefore M$ is a closed subspace of X. Now, $M = X$. Assume not. Then there exists a nonzero element x in X such that 5.3

$$B(x,y) = (x, \mathfrak{z}(y))_X = 0, \quad y \in X.$$

Let $y = x$. Then

$$|B(x,x)| \geq C \|x\|_X^2 = 0.$$

$\therefore M = X$. By Riesz Representation theorem, there exists a unique vector $w \in X$ with

$$f(x) = (x, w)_X, \quad x \in X.$$

Since $X = M$, we can find a vector $y \in X$ with

$$w = \mathfrak{z}(y).$$

$$\therefore f(x) = (x, w)_X = (x, \mathfrak{z}(y))_X = B(x,y), \; x \in X.$$

Suppose there exists another $y_1 \in X$ such that

$$f(x) = B(x,y) = B(x,y_1), \; x \in X$$

Then

$$0 = |B(y-y_1, y-y_1)| \geq C_2 \|y-y_1\|_X^2 = 0$$

$\therefore y = y_1$ and the proof is complete.