Lecture 4

We get

$$2\,\mathrm{Re}(T_\sigma\varphi,\varphi) \ge \delta\|\varphi\|_{0,2}^2 - \|T_{r_6}\varphi\|_{\frac{1}{2},2}\|\varphi\|_{-\frac{1}{2},2},$$

where $r_6 \in S^{-1}$. Since $r_6 \in S^{-1} \subset S^{-\frac{1}{4}}$, by the L^2-boundedness theorem, there exists a positive constant μ with

$$\|T_{r_6}\varphi\|_{\frac{1}{2},2} \le \mu\|\varphi\|_{-\frac{1}{4}+\frac{1}{4},2} = \mu\|\varphi\|_{-\frac{1}{2},2}, \quad \varphi\in\mathcal{S}.$$

$$\therefore\ 2\,\mathrm{Re}(T_\sigma\varphi,\varphi) \ge \delta\|\varphi\|_{0,2}^2 - \mu\|\varphi\|_{-\frac{1}{2},2}^2, \quad \varphi\in\mathcal{S}.$$

But

$$\mu\|\varphi\|_{-\frac{1}{2},2}^2 = \int_{\mathbb{R}^n} \mu\langle\xi\rangle^{-1}|\hat\varphi(\xi)|^2\,d\xi = I + J,$$

where

$$I = \int_{\mu\langle\xi\rangle^{-1}\le\frac{\delta}{2}} \mu\langle\xi\rangle^{-1}|\hat\varphi(\xi)|^2\,d\xi,$$

$$J = \int_{\mu\langle\xi\rangle^{-1}\ge\frac{\delta}{2}} \mu\langle\xi\rangle^{-1}|\hat\varphi(\xi)|^2\,d\xi.$$

But

$$I \le \frac{\delta}{2}\int_{\mathbb{R}^n}|\hat\varphi(\xi)|^2\,d\xi = \frac{\delta}{2}\|\varphi\|_{0,2}^2.$$

Next,

$$\mu\langle\xi\rangle^{-1} \ge \frac{\delta}{2} \Rightarrow \forall s\ge\frac{1}{2},$$

$$\mu\langle\xi\rangle^{-1} = \mu\langle\xi\rangle^{2s-1}\langle\xi\rangle^{-2s} \le \mu\left(\frac{2\mu}{\delta}\right)^{2s-1}\langle\xi\rangle^{-2s}.$$

So, every all $s \geq \frac{1}{2}$,

$$J \leq \mu\left(\frac{2\mu}{\gamma}\right)^{2s-1} \int_{\mathbb{R}^n} \langle\xi\rangle^{-2s} |\hat{\varphi}(\xi)|^2 \, d\xi$$

$$\leq C_s' \, \|\varphi\|_{-s,2}^2, \quad C_s' = \mu\left(\frac{2\mu}{\gamma}\right)^{2s-1}.$$

$$\therefore \; 2 \operatorname{Re}(T_\sigma \varphi, \varphi) \geq \gamma \|\varphi\|_{0,2}^2 - C_s' \|\varphi\|_{-s,2}^2, \quad \varphi \in \mathcal{S}.$$

Strong Solutions of Pseudo-Differential Equations

Let $m > 0$. Consider the pseudo-differential equation

$$T_\sigma u = f \text{ on } \mathbb{R}^n,$$

where $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Let $u \in L^p(\mathbb{R}^n)$ be such that $u \in \mathcal{D}(T_{\sigma,0})$ and

$$T_{\sigma,0} u = f.$$

Then we call u a strong solution of $T_\sigma u = f$ on $\mathbb{R}^n$.

We begin with strong solutions in $L^2(\mathbb{R}^n)$

Theorem: Let $\sigma \in S^{2m}$, $m > 0$, be an elliptic symbol such that there exists a positive constant C for which

$$\operatorname{Re}(T_\sigma \varphi, \varphi) \geq C \|\varphi\|_{m,2}^2, \quad \varphi \in \mathcal{S}.$$

Then for all $f \in L^2(\mathbb{R}^n)$ the pseudo-differential equation $T_\sigma u = f$ on $\mathbb{R}^n$ has a unique strong solution u in $L^2(\mathbb{R}^n)$.

Lemma Under the hypotheses of the theorem, there exists a positive constant C such that
$$\operatorname{Re}(T_\sigma u, u) \geq C \|u\|^2_{m,2}, \quad u \in H^{m,2}.$$

Proof: Let $u \in H^{m,2}$. Then there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ such that
$$\varphi_k \longrightarrow u$$
in $H^{m,2}$ as $k \to \infty$. Since $T_\sigma : H^{m,2} \to H^{-m,2}$ is a bounded linear operator, $\therefore$ there exists a positive constant C with

$$\begin{aligned}
&|(T_\sigma \varphi_k, \varphi_k) - (T_\sigma u, u)| \\
&= |(T_\sigma \varphi_k, \varphi_k) - (T_\sigma u, \varphi_k) + (T_\sigma u, \varphi_k) - (T_\sigma u, u)| \\
&= |(T_\sigma(\varphi_k - u), \varphi_k) + (T_\sigma u, \varphi_k - u)| \\
&= \|T_\sigma(\varphi_k - u)\|_{-m,2} \|\varphi_k\|_{m,2} + \|T_\sigma u\|_{-m,2} \|\varphi_k - u\|_{m,2} \\
&\leq C\Big(\|\varphi_k - u\|_{m,2} \underbrace{\|\varphi_k\|_{m,2}}_{\text{bounded sequence}} + \|u\|_{m,2} \|\varphi_k - u\|_{m,2}\Big)
\end{aligned}$$

$\longrightarrow 0$ as $k \to \infty$

$\therefore \operatorname{Re}(T_\sigma \varphi_k, \varphi_k) \longrightarrow \operatorname{Re}(T_\sigma u, u)$ as $k \to \infty$.

Since $\|\varphi_k\|_{m,2} \longrightarrow \|u\|_{m,2}$ as $k \to \infty$.

$\therefore \operatorname{Re}(T_\sigma u, u) \geq C\|u\|^2_{m,2}, \quad u \in H^{m,2}.$