Lecture 3

Theorem (Gårding's Inequality) Let $\sigma \in S^{2m}$, $m \in \mathbb{R}$, be strongly elliptic. Then we can find a positive constant C' and a real constant C_s for every $s \geq \frac{1}{2}$ such that

$$\mathrm{Re}(T_\sigma \varphi, \varphi) \geq C' \|\varphi\|^2_{m,2} - C_s \|\varphi\|^2_{m-s,2}, \quad \varphi \in \mathcal{S}.$$

Proof: Let $T_\tau = J_m T_\sigma J_m$, where $J_m = T_{\sigma_m}$ and

$$\sigma_m(\xi) = (1+|\xi|^2)^{-m/2} = \langle \xi \rangle^{-m}, \quad \xi \in \mathbb{R}^n.$$

Now, $T_\sigma J_m = T_{\tau_1}$, where $\tau_1 \in S^m$ and

$$(*) \quad \tau_1 - \langle\ \rangle^{-m} \sigma \in S^{m-1}.$$

Then $T_\tau = J_m T_{\tau_1}$, where $\tau \in S^0$ and

$$(+) \quad \tau - \langle\ \rangle^{-m} \tau_1 \in S^{-1}.$$

Multiplying $(*)$ by $\langle\ \rangle^{-m}$ and adding to $(+)$, we get

$$\tau - \langle\ \rangle^{-2m} \sigma \in S^{-1}.$$

We write

$$\tau = \langle\ \rangle^{-2m} \sigma + r_1, \quad r_1 \in S^{-1}.$$

σ is strongly elliptic $\Rightarrow$ there exist positive constants γ and κ Lemma 2
such that

$$\mathrm{Re}\,\sigma \geq \gamma \langle\ \rangle^{2m} - \kappa \langle\ \rangle^{2m-1}.$$

$$\therefore \quad \mathrm{Re}\,\tau = \langle\ \rangle^{-2m} \mathrm{Re}\,\sigma + \mathrm{Re}\, r_1 \geq \gamma - \kappa \langle\ \rangle^{-1} + \mathrm{Re}\, r_1$$
$$= \gamma - \kappa' \langle\ \rangle^{-1},$$

where $\kappa' > 0$. $\therefore$ τ is a strongly elliptic symbol in S^0.

Suppose that Gårding's inequality is valid for all strongly elliptic symbols in S^0. Then we can find a positive constant C' and a real constant C_s for all $s \geq \frac{1}{2}$ such that

$$\begin{aligned} \mathrm{Re}(T_\sigma \varphi, \varphi) &= \mathrm{Re}(J_{-m} T_\tau J_{-m} \varphi, \varphi) \\ &= \mathrm{Re}(T_\tau J_{-m}\varphi, J_{-m}\varphi) \\ &\geq C' \|J_{-m}\varphi\|^2_{0,2} - C_s \|J_{-m}\varphi\|^2_{-s,2} \\ &= C'\|\varphi\|^2_{m,2} - C_s\|\varphi\|^2_{m-s,2}, \quad \varphi \in \mathcal{S}. \end{aligned}$$

We can now prove Gårding's inequality for $m=0$. By Lemma 2, there exist positive constants γ and k such that

$$\mathrm{Re}\,\sigma \geq \gamma \langle\ \rangle^0 - k\langle\ \rangle^{-1},$$

i.e.,

$$\mathrm{Re}\,\sigma + k\langle\ \rangle^{-1} \geq \gamma.$$

Let $F \in C^\infty(\mathbb{C})$ be such that

$$F(z) = \sqrt{\frac{\gamma}{2} + z}, \quad z \in [0,\infty).$$

Let

$$\begin{aligned} \tau(x,\xi) &= F\left(2\left(\mathrm{Re}\,\sigma(x,\xi) + k\langle\xi\rangle^{-1} - \gamma\right)\right) \\ &= \sqrt{2\,\mathrm{Re}\,\sigma(x,\xi) + 2k\langle\xi\rangle^{-1} - \frac{3}{2}\gamma}, \quad x,\xi \in \mathbb{R}^n. \end{aligned}$$

Now, $T_\tau^* = T_{\tau^*}$, where $\tau^* \in S^0$ and

$$\tau - \tau^* = r_2 \in S^{-1}.$$

Also, $T_\tau^* T_\tau = T_\lambda$, where $\lambda \in S^0$ and

$$\lambda - \tau^*\tau = r_3 \in S^{-1}.$$

$$\therefore \quad \lambda - (\tau - r_2)\tau = r_3$$

$$\therefore \quad \lambda = 2\,\mathrm{Re}\,\sigma + 2\kappa\langle\xi\rangle^{-1} - \frac{3}{2}\gamma + r_4, \quad r_4 \in S^{-1}.$$

$$= 2\,\mathrm{Re}\,\sigma - \frac{3}{2}\gamma + r_5, \quad r_5 \in S^{-1}.$$

$$\therefore \quad \sigma + \bar{\sigma} = \lambda + \frac{3}{2}\gamma - r_5.$$

$$\therefore \quad \sigma + \sigma^* = \lambda + \frac{3}{2}\gamma - r_6, \quad r_6 \in S^{-1}.$$

But for all $\varphi \in \mathcal{S}$,

$$(T_\lambda \varphi, \varphi) = (T_\tau^* T_\tau \varphi, \varphi) = (T_\tau \varphi, T_\tau \varphi) \geq 0, \quad \varphi \in \mathcal{S}.$$

$$\therefore \quad 2\,\mathrm{Re}(T_\sigma \varphi, \varphi) = (T_\sigma \varphi, \varphi) + (T_\sigma^* \varphi, \varphi)$$

$$= (T_\lambda \varphi, \varphi) + \frac{3}{2}\gamma(\varphi, \varphi) - (T_{r_6}\varphi, \varphi)$$

$$\geq \gamma \|\varphi\|_{0,2}^2 + \left\{\frac{1}{2}\gamma\|\varphi\|_{0,2}^2 - \|T_{r_6}\varphi\|_{\frac{1}{2},0}\|\varphi\|_{-\frac{1}{2},0}\right\}, \quad \varphi \in \mathcal{S}.$$

To be continued and completed tomorrow.