Lecture 2

Theorem: (Gårding's Inequality) Let $\sigma \in S^{2m}$, $m \in \mathbb{R}$, be strongly elliptic. Then we can find a positive constant C' and a real constant C_s for all $s \geq \frac{1}{2}$ such that

$$\mathrm{Re}(T_\sigma \varphi, \varphi) \geq C' \|\varphi\|^2_{m,2} - C_s \|\varphi\|^2_{m-s,2}, \quad \varphi \in \mathcal{S}.$$

Lemma 2: Let $\sigma \in S^{2m}$, $m \in \mathbb{R}$, be strongly elliptic. Then there exist positive constants γ and K such that

$$\mathrm{Re}\,\sigma(x,\xi) \geq \gamma \langle \xi \rangle^{2m} - K \langle \xi \rangle^{2m-1}, \quad x, \xi \in \mathbb{R}^n,$$

where $\langle\ \rangle$ is the function on $\mathbb{R}^n$ given by

$$\langle \xi \rangle = (1+|\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n.$$

Remark

There exist positive constants C_1 and C_2 such that

$$C_1(1+|\xi|^2)^{1/2} \leq (1+|\xi|) \leq C_2(1+|\xi|^2)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^n.$$

To see this,

$$(1+|\xi|^2)^{1/2} \leq (\max(1,|\xi|^2))^{\frac{1}{2}} \leq (1+|\xi|), \quad \xi \in \mathbb{R}^n.$$

Next,

$$(1+|\xi|)^2 = 1+2|\xi|+|\xi|^2 \leq 1+1+|\xi|^2+|\xi|^2)$$
$$= 2(1+|\xi|^2)$$
$$\Rightarrow (1+|\xi|) \leq 2^{\frac{1}{2}}(1+|\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n.$$

Proof of Lemma 2:

Proof: By strong ellipticity, we can find positive constants C and R such that

$$\mathrm{Re}\,\sigma(x,\xi) \geq C\langle \xi \rangle^{2m}, \quad |\xi| \geq R.$$

Since $\sigma \in S^{2m}$, we can find a positive constant K such that

$$|\sigma(x,\xi)| \leq K\langle \xi \rangle^{2m}, \quad x, \xi \in \mathbb{R}^n.$$

$\therefore \ |\mathrm{Re}\, \sigma(x,\xi)| \le |\sigma(x,\xi)|$

$$\le K\langle\xi\rangle^{2m} \le K(1+R^2)^m, \ |\xi| \le R.$$

$\therefore$ there exists a positive constant M such that

$$\mathrm{Re}\, \sigma(x,\xi) \ge -M, \ |\xi| \le R.$$

Now,

$\dfrac{\mathrm{Re}\, \sigma}{\langle \ \rangle^{2m-1}}$ is continuous on $\{\xi \in \mathbb{R}^n : |\xi| \le R\}$.

$\therefore$ there is a positive constant k such that

$$\frac{\mathrm{Re}\, \sigma(x,\xi)}{\langle\xi\rangle^{2m-1}} > -k, \ |\xi| \le R.$$

$\therefore \ \mathrm{Re}\, \sigma(x,\xi) + k\langle\xi\rangle^{2m-1} > 0, \ |\xi| \le R.$

Now, $\dfrac{\mathrm{Re}\, \sigma + k\langle \ \rangle^{2m-1}}{\langle \ \rangle^{m}}$ is a positive and continuous function

on $\{\xi \in \mathbb{R}^n : |\xi| \le R\}$, $\therefore$ there is a positive constant δ such that

$$\frac{\mathrm{Re}\, \sigma(x,\xi) + k\langle\xi\rangle^{2m-1}}{\langle\xi\rangle^{2m}} \ge \delta, \ |\xi| \le R.$$

So, the lemma is proved with

$$\gamma = \min(C, \delta).$$

Proof of Garding's Inequality

Let $T_\tau = J_m T_\sigma J_m$, where

$$J_m = T_{\sigma_m},$$

where $\sigma_m(\xi) = (1+|\xi|^2)^{-m/2}$, $\xi \in \mathbb{R}^n$.

So,

$$T_\sigma J_m = T_{\tau_1}$$

where $\tau_1 - \langle\ \rangle^{-m} \sigma \in S^{m-1}$

Now

$$T_\tau = J_m T_{\tau_1}$$

and $\tau - \langle\ \rangle^{-m} \tau_1 \in S^{-1}$.

$\therefore\ \tau - \langle\ \rangle^{-2m} \sigma \in S^{-1}$

$\therefore\ \tau = \langle\ \rangle^{-2m} \sigma + r,\ r \in S^{-1}$

$$\begin{aligned} \operatorname{Re} \tau &= \langle\ \rangle^{-2m} \operatorname{Re} \sigma + \operatorname{Re} r \\ &\geq \langle\ \rangle^{-2m} \gamma \langle\ \rangle^{2m} - k\langle\ \rangle^{-1} + \operatorname{Re} r \\ &\geq \gamma - k' \langle\ \rangle^{-1}. \end{aligned}$$

$\therefore\ \tau$ satisfies the conclusion of Lemma 2 with $m = 0$.