Lecture 17

Theorem: Let X, Y, Z be complex, separable and infinite-dimensional Hilbert spaces. Let $A_1 : X \to Y$ and $A_2 : Y \to Z$ be Fredholm operators. Then $A_2 A_1 : X \to Z$ is a Fredholm operator and

$$i(A_2 A_1) = i(A_2) + i(A_1).$$

Proposition Let X and Y be complex and separable Hilbert spaces. Let $A : X \to Y$ be a Fredholm operator. Then

$$Y = N(A^t) \oplus R(A). \text{ (Ex. 20.9)}$$

Proof of Theorem: Let $M_1 = R(A_1) \cap N(A_2)$, a subspace of Y and $\dim M_1 < \infty$. Write

$$R(A_1) = M_1 \oplus M_2, \quad N(A_2) = M_1 \oplus M_3,$$
$$Y = R(A_1) \oplus M_3 \oplus M_4.$$

So, $\dim M_j < \infty$, $j = 1, 3, 4$ and M_2 is a closed subspace of Y. Let

$$X_1 = N(A_2 A_1) \ominus N(A_1).$$

Claim: $A_1(X_1) = M_1$.

Proof: Let $x_1 \in X_1$. Then $A_1 x_1 \in R(A_1)$ and

$$x_1 + \omega \in N(A_2 A_1), \quad \omega \in N(A_1)$$

$$\therefore \quad (A_2 A_1)(x_1 + \omega) = A_2 A_1 x_1 = 0$$

$\therefore A_1 x_1 \in N(A_2)$. $\therefore$ $A(X_1) \subseteq M_1$.

Conversely, let $m_1 \in M_1$. Then $\exists\, x \in X$ such that

$$m_1 = A_1 x.$$

Now,

$$A_2 m_1 = A_2 A_1 x = 0.$$

Write

$$x = x_0 - \omega, \quad x_0 \in N(A_2 A_1), \quad \omega \in N(A_1).$$

$\therefore\ m_1 = A_1 x_0 - A_1 \omega = A_1 x_0$. $\therefore\ m_1 \in A(X_1)$.

$\therefore\ A_1(X_1) = M_1$. $\therefore\ X_1 \underset{\text{bijection}}{\simeq} M_1$.

Let $Z_4 = R(A_2) \ominus R(A_2 A_1)$. Then

Claim $Z_4 = A_2(M_4)$, i.e., $Z_4 \underset{\text{bijection}}{\simeq} M_4$

Proof: For all $y \in Y$,

$$A_2 y = A_2(A_1 x_0 + m_3 + m_4), \quad x_0 \in X,\ m_3 \in M_3,\ m_4 \in M_4.$$

$$= A_2 A_1 x_0 + A_2 m_4$$

$$= A_2 m_4 + A_2 A_1 x_0$$

$\therefore$ if we let $d_j = \dim M_j$, $j = 1, 3, 4$, then

$$i(A_2 A_1) = \dim N(A_2 A_1) - \dim N((A_2 A_1)^t)$$

$$= \dim N(A_2 A_1) - \dim(Z \ominus R(A_2 A_1))$$

$$= \dim N(A_1) + d_1 - \dim(Z \ominus R(A_2)) - d_4$$

Also,

$$i(A_1) + i(A_2)$$

$$= \dim N(A_1) - \dim N(A_1^t) + \dim N(A_2) - \dim N(A_2^t)$$

$$= \dim N(A_1) - \dim(Y \ominus R(A)) + \dim N(A_2) - \dim(Z \ominus R(A_2))$$

$$= \dim N(A_1) - \cancel{d_3} - d_4 + d_1 + \cancel{d_3} - \dim(Z \ominus R(A_2))$$

$\therefore\ i(A_2 A_1) = i(A_1) + i(A_2)$.

To compute indices, we use trace class operators.

Let X be a complex and separable Hilbert space

Let $A : X \to X$ be a compact operator. Then

$A^t A : X \to X$ is compact and positive, i.e.,

$$(A^t A x, x) \geq 0, \quad x \in X. \text{ (Why?)}$$

$\therefore \sqrt{A^t A} : X \to X$ is compact and positive.

$\therefore$ there exists an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ for X

consisting of eigenvectors of $\sqrt{A^t A}$.

For $k = 1, 2, \cdots$, let s_k be the eigenvalue of

$\sqrt{A^t A}$ corresponding to φ_k.

- Suppose that $\sum_{k=1}^{\infty} s_k < \infty$. Then we call

  $A : X \to X$ a <u>trace class operator</u>.

- Suppose that $\sum_{k=1}^{\infty} s_k^2 < \infty$. Then we call

  $A : X \to X$ a <u>Hilbert - Schmidt operator</u>.